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Jean Cavaillès in the legacy of Léon Brunschvicg: mathematical philosophy and the problems of history[(*)]

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Summary. *The philosophy of the history of mathematics expounded by Cavaillès bears a close and contrasting relationship to that set out by Brunschvicg in La Modalité du Jugement.* (1897). *The activity of scientific judgement is said to be* mixed, *between* (*ideal*) *judgements of interiority and* (*realistic*) *judgements of exteriority. The mixed form of the historical activity of knowledge is the modality of the* possible. *Hence a historical epistemology that claims Kantian idealist filiation and rejects speculative idealism. Cavaillès, a thinker of the creative necessity of mathematical development, reduces the role of intellectual adventure and possibility in its history, and in so doing distances himself from a master to whom Canguilhem would have remained closer. The recent history of mathematics, while paving the way for Kronecker's revenge on the abstract set-theoretical conceptions that inspired Cavaillès's necessitarianism, leads us to reconsider the radicalism that opposed him to Brunschvicg's philosophy of the modality of judgement.*

> Even if, as Hegel says, thought is ungrateful, it is not bound to be unfair to thinkers. Even if mathematical philosophy may seem to some not to address the full range of genuinely philosophical problems, none of his students can forget what he owes to Léon Brunschvicg, just as none of those who knew him can forget, without bitterness, the tragic death of Jean Cavaillès.[(1)]

[(*)] Alain Michel, « Jean Cavaillès dans l'héritage de Léon Brunschvicg : la philosophie mathématique et les problèmes de l'histoire », *Revue de métaphysique et de morale*, n° 105, p. 9-36 © Presses Universitaires de France / Humensis, 2020. The editors would like to thank Denis Kambouchner and the editorial board of the *Revue de métaphysique et de morale*, for their kind authorisation to translate and publish this text.

[(1)] Georges Canguilhem, "Note sur la situation faite en France à la philosophie biologique", Revue de métaphysique et de morale, vol. 52, no. 3-4, juillet-octobre 1947, p. 324; Œuvres complètes, Paris, 2015, vol. IV, p. 309.

To understand, Cavaillès said of mathematical methods, is to catch the gesture, and to be able to continue. To talk about Cavaillès is first and foremost, and should be above all, to reflect on the possible ways of continuing after him. Of course, it is first and foremost about history, and the history of mathematics has since been written according to its own standards, often diverging significantly from those that could have been drawn from his work. But it is also a question of philosophy, of mathematical philosophy, for an author who never thought it possible to separate these two types of questioning. What can be retained of the terms in which Cavaillès philosophically thought about mathematics, particularly in its relationship with history? Under what conditions, and with what consequences for our way of writing the history of mathematics? Needless to say, we will only offer a few remarks here, as it is clear that such an undertaking requires the contribution of all those mathematicians, philosophers, and historians of mathematics or science who still find interest and stimulation in reading his writings today.

The first obvious fact is that there is a historical distance between us and Jean Cavaillès. This has at least two directly visible consequences.

Firstly, the lively proximity brought about and long maintained by the presence around us of masters who not only knew and rubbed shoulders with him, but shared a history, not to say a destiny, with him, this kind of complicity has disappeared. It seems that, at the same time, we are now situated, with regard to him and his work, at the same reflective distance as that which separates us from the great works of the philosophical tradition — even taking into account its dramatic interruption, the cause of an incompleteness that has not always been spared in the work of philosophers from less barbaric times.

Then there was the natural and legitimate development of criticism, both from mathematicians who, speaking from within the practice of mathematics, were disappointed by the relatively limited openness of Cavaillès's thinking to the diversity of its fields[(2)], and from philosophers of logic, anxious to distance themselves from the legacy of a certain rationalist idealism, a common belief — never denied — shared by the masters he had come to identify with. While they could accept the assertion of the rational power

[(2)] Often, in a fruitful comparison with his friend Albert Lautman, who had also been a student of Brunschvicg.

of mathematics, they found it more difficult to accept the counterpart of obstinate mistrust towards developments in formal logic, which was always suspected of favouring the revival of scholastic Aristotelianism.

In short, Jean Cavaillès has almost become for us a philosopher like any other. We are reduced, and in any case will be increasingly so, to looking to his writings for the elements of examination or judgement. Among the possibilities that such a situation allows is the option of dilating the space of our critical appreciation accordingly. This dilation can take place in two directions, by going backwards, towards the sources, and by projecting forwards, towards his posterity, which is still very much alive in France. On the one hand, it may be useful to try to trace the origins of some of his fundamental ideas and thus, in order to understand him, to establish their history. The time has come to make a history of the philosophy, or what might have been the philosophy, of Cavaillès[(3)]. On the other hand, it is always legitimate, and can be instructive, to extend the examination of the historical theme further, into our present day, in order to evaluate by comparison the value and significance of the foundations that it assigned, explicitly or implicitly, to historical practice.

To say that the existence of Cavaillès today tends to be reduced to that of his written work is to recognise that consideration of the latter must, for the most part, impose its demands on commentary. The first is to maintain a balance between its two main sources of inspiration, philosophy and mathematics. This is why we will first of all try to reconstruct the elements of the philosophical tradition that seem to us to have been most important to him, in an attempt to assess their value for the history of mathematics. Various accounts, more or less anecdotal but concordant[(4)], attest to his desire to promote an authentic philosophy of science in France after the war, based on a solid foundation of reflection on the mathematics of the time. For him, authentic meant first and foremost independence, in particular from the positions taken by the practitioners of mathematical science themselves, who were also his natural and direct

[(3)] We already have, of course, the articles and works of H. BENIS SINACEUR, in particular *Jean Cavaillès. Philosophie mathématique*, Paris, Puf, 1994; *Cavaillès*, Paris, Les Belles Lettres, 2013. We will confine ourselves here to mentioning the themes that situate Cavaillès in the Brunschvicgian lineage, and which we feel are essential to a proper understanding of his work.

[(4)] We are thinking, among others, of G. Bachelard, G. Canguilhem, R. Aron.

interlocutors. These same testimonies, and some of his writings, show him to be quite scrupulous on this point[5]. However, this claim to independence for philosophy can only be substantiated by drawing on its past, which has been constituted into history. We have enough of his texts to try to specify, if not what could have formed the precise content of his "theory of science", at least what were its sources, its inspiration, everything that makes it part of what we call a tradition. And, through the many signs that the texts give us of the domination of the Kantian tradition[6], it is not difficult to see the origin of this domination: the teaching of Léon Brunschvicg. We will therefore first examine what was essentially the main topics addressed by Cavaillès: science, mathematics, philosophy, history, and the lesson of Brunschvicg.

Brunschvicg was responsible for introducing in France a philosophical doctrine in which history and philosophy were indissolubly linked[7]. We will try to identify the elements that seem most significant, starting with Brunschvicg's first philosophical work, his main thesis, *La Modalité du jugement* (*The Modality of Judgement*), first published in 1897, a work that contains the seeds of almost all the themes that would later be developed.

[5] To see this, it is sufficient to consult the collective notice, apparently mainly the work of Cavaillès and Lautman, written for the collection "Essais philosophiques" created by Cavaillès and Aron, published by Hermann, of which Lautman's *Nouvelles recherches sur la structure dialectique des mathématiques* formed the first fascicule: "Any attempt at clarification that cannot currently be satisfied by scientific techniques, or cannot be satisfied by them in their normal development, is philosophical [...] In the contemporary system of concepts and processes of authentic thought, philosophers have their own specific role [...] This does not mean subordination to science or renunciation of fundamental questions of metaphysics or moral philosophy [...]." (see the collection of writings by A. LAUTMAN, *Les Mathématiques, les idées et le réel physique*, Paris, Vrin, 2006, p. 235).

[6] We have no doubt, for example, that it is only on the basis of Kantian doctrine that we will be able to build up the 'theory of reason' that Cavaillès calls for at the end of the first chapter of *Méthode axiomatique et formalisme* (*Œuvres complètes de philosophie des sciences*, Paris, Hermann, 1994, p. 29; we will cite all Cavaillès's texts in this volume, called *Œuvres*, and in the pagination appropriate to it). A whole series of texts deals directly with Kantian themes: in particular, a large part of the introduction to this last work (§ 2, c: schématisme et intuition spatiale chez Kant, *Œuvres*, pp. 34-40), the remarks on intuition which conclude *Transfini et continu* (*Œuvres*, pp. 469-472), and the first part of the posthumous work, which examines the meaning and scope, for the theory of science, of the Kantian notion of form.

[7] "The doctrine of Léon Brunschvicg", writes M. Gueroult, "is the type of philosophy in which history and philosophy are so closely interwoven that they can scarcely be distinguished from one another" (*Histoire de l'histoire de la philosophie*, t. III, p. 881).

Right at the start of Chapter II, the author clarifies the historical significance of the issue at hand. Stressing the need for reflection on history, he makes it clear that this cannot be reduced to idle curiosity. A philosophical problem is never given from the outside, imposed from the outside by external circumstances. Since it exists only in the mind, no objective criterion can be found to ensure that it is not fiction or artifice, but has a real foundation in the nature of philosophical speculation. Such a criterion can only be provided by examining the various doctrines that have arisen in history: "it is only in this way that our individual meditation emerges to some extent from its inevitable isolation and communes with the thought of humanity". In this sense, history is the permanent support of philosophical meditation. We shall see later that it is much more than that.

What is philosophy for Léon Brunschvicg? "Intellectual activity becoming aware of itself [...] that is what philosophy is all about"[(8)]. Philosophy stems entirely from this return of reflection — hence the reference to critical thought — to intellection as a spontaneous or natural activity. It is a position of immanence. It is explicitly asserted by Cavaillès, at least in his theses[(9)]. How can we conceive of this activity? By going back to the act that the mind performs in the concept. This act is the judgement, which posits the copula, brings duality back to unity, and remains the same through the diversity of its logical expressions. Of this judgment of the mind, the concept is the condensed expression, reasoning the developed expression. But judgement itself is the complete and unique act of intellectual activity, the beginning and the end of the mind[(10)].

[(8)] *La Modalité du jugement*, 2e éd., Paris, Alcan, 1934, p. 5.

[(9)] Cavaillès consistently uses this theme of activity in his writings, without the assignment of its subject always being perfectly clear, nor, above all, uniform. But, as with other categories of "idealistic" philosophical origin, we note, in relation to Brunschvicg, a visible shift in its meaning in a positive, material, almost corporeal sense. For example, Cavaillès readily refers to the "gestures" of the theorist to describe the generation of concepts or methods, for example at the end of *Méthode axiomatique et formalisme*, *Œuvres*, pp. 186-187: "every abstract object [...] is a gesture on a gesture [...] on a gesture on the primitive sensible", or again in *La Pensée mathématique*, the text of his talk to the Société française de philosophie on 4 February 1939, *Œuvres*, p. 602: "the mathematical object is thus [...] always correlative of gestures accomplished by the mathematician in a given situation".

[(10)] *La Modalité du jugement*, op. cit. p. 24: "the study of intellectual activity that defines philosophy is therefore a study of judgment [...] judgment is the complete

The philosophical problem is to find the reason for this judgement. Plato and Descartes, each in their own way, looked for it in a transcendent principle; Kant had the merit of looking for it in the analysis of human thought, in the *Critique*. This discovery gave philosophy its definitive form, and, as a result, it is the obligatory starting point for philosophical reflection: it is from Kant that we must begin to philosophise. But Kantianism cannot be taken literally, which would mean reducing it to an object of study for historians of philosophy. If Kantianism can be given the weighty task of stimulating philosophical reflection from the outset, it is on condition that we rediscover, in the texts and beyond the interpretations, the true, authentic Kantianism that is a method, not a system.

It is imperative to rediscover the method that Kant was forced to adopt in the *Critique of Judgment*, where, in order to deal with aesthetic and teleological judgment, he had to accept that judgment could not be reduced to a concept. Critical reflection then finds a privileged terrain in the treatment of the problem of *modality*. Modality does not, in fact, belong to the judgment considered in its spontaneous expression, but is added to it by the mind: it is due entirely to critical reflection. It is a judgment about judgment. By undermining the foundations of Cartesian philosophy, the refutation of the ontological argument led to a decisive advance: the problem of modality ceased to be a theological problem and became a human problem. At the same time, it enables reflection on human knowledge, whose characteristics, in modern thought, can only be provided by science.

Léon Brunschvicg distinguishes three forms of judgement, three "absolute acts of judging": the judgement of interiority, the judgement of exteriority, and the mixed judgement. The first form develops relationships that are immanent to ideas, whose unity lies in their mutual interiority. The copula "is" signifies this primary unity, the originating source of truth. In the second, judgement posits being, not as an intrinsic relationship between ideas, but on the contrary as the exclusion of all interiority, as pure exteriority. The relation of the subject who judges to the object he judges is absolute heterogeneity: it is explained by the shock of the given, which signifies the irreducibility of the mind to something other than itself. Abstract analysis thus leads to dualism, to a double meaning of the copula "is", signifying either thought or being, and the two

act of intellectual activity and the unique act [...] the beginning and the end of the mind, [...] the mind itself".

being absolutely opposed as affirmation and negation. The duality of being and thought is primitive and irreducible, and reason cannot conceive of their unity. Here we find the essential contribution of transcendental philosophy. Interiority is an affirmation of the mind: judgement establishes the immanent relationships to ideas. Exteriority is its absolute negation: judgement posits the being as irreducibly heterogeneous, external. But negation is not the absence of affirmation, it is the act of judgement. As such, it implies a positive foundation. The reason for a negation is not the absence of a reason to exist, but the presence of a real opposition. The union of affirmation and negation in one and the same act of the mind, imposed by the fact of knowledge, is in itself incomprehensible; it can only be in judgment a *mixed*, equivocal form, capable of participating in two radically distinct orders of principles. If knowledge, the source of truth, were perfectly transparent to the mind, there could never be any doubt about its reality. Those who are in error would never believe themselves to be in possession of the truth. But this is not the case: spontaneous knowledge is equally susceptible to truth and error. Knowledge that is perpetually threatened with being false, unstable and precarious by nature is only possible knowledge. The mixed form is the modality of possible knowledge. It is essentially that of science[(11)], whose ceaseless progress implies a perpetual questioning of the principles taken for granted, a continuous exploration over which doubt presides. Hence what will henceforth be the framework for the development of Brunschvicg's philosophy: that of the mixed category, possibility.

It seems to us that this question of the status of historical judgement, with the choice of methods and the hierarchy assigned to them, can be used as a touchstone to distinguish between the different orientations of the two philosophies of the master and the disciple, even if one of them was still only in the making, and will remain so forever. We will attempt to show how, in a departure that seems to us to be assumed with regard to his master, Cavaillès was only able to emphasise the role of necessity in mathematical development, which is not exclusive to that recognised in contingency, by reducing the role of possibility accordingly. Even if this were proven to be true, the task of understanding the profound reasons behind it would still remain.

[(11)] "The modality of scientific judgement is possibility", op. cit. p. 164 (see also p. 52).

The choice of possibility as categorical modality for the history of science enables Brunschvicg to conceptualise this discipline according to the norm of reflexive thought. Consequently, it is distinguished from simple empirical research, which is conducted for the needs of a particular purpose, at the whim of commemorative events or the search for titles of priority. In this sense, history can be considered a manifestation of philosophical rationality. Brunschvicg, in his major works, in particular *Les Étapes de la philosophie mathématique* [*The Stages of Mathematical Philosophy*] (1912) and *L'Expérience humaine et la causalité physique* [*Human Experience and Physical Causality*] (1922), sought to rediscover the fundamental principles of intellectual activity by examining the history of science. The stages of scientific progress are also stages in the progress of knowledge, that is, in the relentless effort of the human mind to increasingly subject externality to the interiority of rational thought, through a series of trial and error, provisional equilibriums, and revolutions[(12)].

It should be added that, in this development of scientific reason, a special place is reserved for mathematics. This is "mathematism", which we find again, brought to a higher degree of incandescence, in Jean Cavaillès. Brunschvicg believed, perhaps influenced by Cournot, that the history of mathematics provides the key to philosophy and its history. He also believed that the development of mathematics is a prerequisite for the development of science. "The consideration of mathematics", he wrote in *Les Étapes...*, "is at the basis of the knowledge of the mind as it is at the basis of the sciences of nature, and for the same reason: the free and fertile work of thought dates from the time when mathematics came to bring to man the true standard of truth"[(13)]. It is his doctrine of the modality

[(12)] Brunschvicg seems to have taken from Cournot (*Matérialisme, Vitalisme, Rationalisme, Étude sur l'emploi des données de la science en philosophie*, 1873, p. 371, quoted in L. BRUNSCHVICG, *L'Expérience humaine et la causalité physique*, p. XIII, and M. GUEROULT, *Histoire de l'histoire de la philosophie*, t. III, p. 881) the idea that scientific crises are "the driving principle behind the truly renovating crises of philosophy". Science progresses through a series of revolutions that call principles into question and promote new procedures. We know that this idea would also animate the epistemological work of Bachelard, himself a faithful admirer of Brunschvicg.

[(13)] *Les Étapes de la philosophie mathématique*, Paris, p. 577. It should be remembered that in 1893, in the anonymous introductory text to the first issue of the *Revue de Métaphysique et de Morale*, probably written by Alphonse Darlu, who had been the teacher of Brunschvicg and Xavier Léon (*La Modalité du jugement* is dedicated to him), it was stated that: "The journal has a marked predilection — in

of judgement that enables Brunschvicg to go beyond the simple position, mentioned above, of history as the support of philosophical meditation, towards a more dynamic conception of a philosophy of reflection that finds its natural material in the history of human thought. Philosophy, having become aware of intellectual activity as a perpetual effort to gradually absorb the judgement of exteriority and thus achieve unity in knowledge, discovers the dynamism of reason. This rational dynamism cannot be known directly, but only in its products, through its works. It can only be grasped by reflecting on the various scientific concepts, linked to their philosophical interpretations, as they have manifested themselves in the course of history. History is the only means available to the philosopher to accomplish his task: "to understand the mind", according to Léon Brunschvicg, "to construct a theory of reason", according to Jean Cavaillès.

This is where Léon Brunschvicg parts ways with German philosophy, and with those of its representatives who might be called post-Kantians. According to him, the latter failed to nurture the new idealism whose seeds were latent in the *Critique*. Brunschvic idealism was to oppose German idealism with what M. Gueroult called a "spiritual positivism"[(14)]. The latter sets itself the exclusive task of understanding the living spirit, of following man in the slow conquest of his mind, the stages of which are prepared and marked by science. It is through science that we discover the works of reason, and it is by starting with scientific works that we can discover the rational approaches that gave rise to them, the whole evolution that Brunschvicg calls the "progress of consciousness". But instead of studying the works of the human mind in order to distinguish

remembrance, if you will, of Plato and Descartes — a predilection as an elder sister, we would say rather, for the mathematical sciences, that great art with inexhaustible resources, born, too, of the human mind" (*Revue de Métaphysique et de Morale*, 1893, Introduction, p. 3). — Cavaillès would certainly not have contradicted these remarks, but goes further in the same direction. Mathematics is the discipline that can teach us what thinking is. See *La Pensée mathématique*, *Œuvres*, p. 625: "I am not trying to define mathematics, but, by means of mathematics, to find out what it means to know, to think. Mathematical knowledge is central to knowing what knowledge is. I don't know what it means to know the real world other than to do mathematics on the real world."

[(14)] "Léon Brunschvicg et la philosophie allemande", *Bulletin de la Société française de philosophie*, LXIV,n° 1, 1970; reprinted in *Études de philosophie allemande*, Hildesheim-New York, 1977, p. 327. The author had used the analogous expression "rational positivism" to characterise Cartesian idealism: see *Descartes selon l'ordre des raisons*, Paris, Aubier, 1953, t. I, p. 384.

the fundamental laws of its activity and the essential relationships gradually revealed by the progress of science, post-Kantians have been preoccupied with the problem of origin, and with establishing filial relationships between these laws. But this is an insoluble problem, just as much as the "ontological" problem. On this last point, it is worth noting, without going into further detail, Cavaillès' unwavering adherence to this position. The conception of the mind as an essential activity inherently implies the elimination of ontology or metaphysics[15]. The texts in which Cavaillès rejects, as an abdication of thought, any projection of it into an ontological absolute (what he most readily calls an 'in itself') are numerous and unambiguous[16], and their author fully accepts the consequences drawn from them by his master. If it has to deprive itself of the ontological refuge, all that remains for thought to constitute itself as mathematics and science is history.

As for the demand for origin, in Léon Brunschvicg's view this is a real deviation from idealist thought. Thus, we see thinkers such as Fichte, Schelling, and Hegel no longer content themselves with deducing the world from knowledge, but also undertaking to deduce nature, history, etc., so that the sequence of events appears as the mark, or even the ratification, of logical necessity. We then move from criticism to meta-criticism, tending towards an absolute rationalism that would subvert the true meaning of Kantian doctrine. And here again, Cavaillès, by refusing to include the problem of the derivation of mathematics from something other than itself within the scope of the tasks of a mathematical philosophy, to the point of deliberately ignoring considerations of psychological or sociological genesis, can be considered not only to have found himself in agreement with the Brunschvicgian abstention, but to have given it a more radical form.

The most representative moment of mathematism, in Brunschvicg's eyes, is the Cartesian moment. It is presented as the very conquest of the autonomy of the mind, converting the spatial and the

[15] "We will take up", writes Brunschvicg, the slogan that Émile Boutroux proposed to philosophy in 1874 in his thesis *De la contingence des lois de la nature*: "It is the act that implies the essence, far from the essence being able to explain the act. It is therefore not the nature of things that should be the supreme object of our scientific research, but their history" (*L'Esprit européen*, Neuchâtel, La Baconnière, 1947, p. 8).

[16] Without claiming to exhaust the subject, see for example *Œuvres*, pp. 472, 503, 505, 517-518, 521, 603, 649, 663. We also know that this is a point of difference with Lautman.

imaginative into rational relationships, reducing exteriority into interiority, manifesting the constructive power of "analysis" — this being taken in the new algebraic sense, which allows Brunschvicg to speak of the "spiritualisation of algebra". One of Descartes' most original conceptions, already proposed in the *Regulae...*, is that of intuition as an act of the mind, the immediate intelligence of a relation, akin to an equation. The ultimate consequence of this is the elimination of any difference in nature between intuition and deduction[17]. "Intuition", he says in *Les Étapes...*, "is not a higher form of representation through which the mind communicates with a thing in itself, and affirms the transcendent reality of the object: it is pure intellection which brings together in an indivisible act of connection a diversity of distinct ideas and affirms their unity as self-evident truth". This is not "a metaphysical faculty", but "the principle of a science that has reached its highest degree of clarity and intelligibility", allowing the intellectualism of modern thought to unfold with unlimited freedom and fertility.

This Cartesian conception is notably what enables Spinoza to contrast sensible intuition, as a receptive faculty whose content is images, with the idea, as an act of the mind. For Spinoza, "intuitive science is self-sufficient; it is merely the development of the internal dynamism that is the nature of thought, the mark of spiritual automatism". Every idea affirms itself, produces its own consequences, and the verification of the validity of knowledge is nothing other than the awareness of the synthetic power of ideas[18].

Cavaillès's references to Spinoza are well known and have often been commented on. As such, they are rather implicit, linked to the choice of certain characteristic expressions, such as "singular essences", and, rather than a pure and simple desire to adopt

[17] For example, among others: "Cartesian intuition is, or tends to be, an intuition not of things but of thought. The reality of thought consists in an act. This act is first and foremost the act of judging. [...] Deduction merely makes intuition explicit". From this "dynamism of intuition", Descartes made "the guiding motif of his physics and metaphysics...", "La pensée intuitive chez Descartes et les cartésiens", *Écrits philosophiques*, t. I, Paris, Puf, 1951, pp. 56-57.

[18] *Les Étapes de la philosophie mathématique*, Paris, new edition, 1972, pp. 140-142. Or again, among others: "according to Spinoza, science is constituted, and it constitutes reality, by the development of an activity which is inherent in the idea conceived as self-affirmation and which, spontaneously, from synthesis to synthesis, extends to the total system of knowledge" ("Sommes-nous spinozistes?", *Écrits philosophiques*, t. I, op. cit., p. 159). — For a critical analysis of the Brunschvicgian comparison between Cartesian intuition and Spinozist intuition, see M. GUEROULT, *Spinoza*, t. II, Paris, Aubier, 1974, pp. 481-482.

them, it would probably be better to speak of a general, principled adherence to Spinozism. Among other things, this includes his oft-reaffirmed conviction of the dominance of necessity in history, and first and foremost, of course, in the history of mathematics[(19)]. Cavaillès developed an original conception of mathematical necessity as immanent to history, inspired by the contemporary development of set theory. We will have to return to this point later, but it is worth emphasising here that this focus on necessity is one of the areas in which Cavaillès differs from Brunschvicg's ideas, which, as we have seen, associate the concept of the history of science with the modality of possible knowledge. "The solution", he says, "that Spinoza will give — implicitly, perhaps, but very clearly nonetheless — to the problem of modality, will be to reduce all the forms of modality to a single type, which is necessity", or again: "The problem of the modality of judgement thus consists for Spinoza in the following alternative: 'there is no middle ground between the possible and the necessary; the possible resolves itself

[(19)] We think in particular of R. Aron in his preface to the 1962 reprint of several of Cavaillès's works: "Jean Cavaillès, from what I remember of our last conversations, insisted on the necessity that governed practical imperatives as well as scientific propositions, 'I am a Spinozist' he said, 'one must resist, fight, face death'" (*Philosophie mathématique*, Paris, Hermann, 1962, p.. 14), to those of Georges Canguilhem, who refers to Cavaillès's Spinozism in almost every one of his speeches (see *Œuvres*, pp. 673, 680, 685-686), and confided in 1947 that between two possible interpretations, one Hegelian, the other Spinozist, of the "dialectic" of the last Cavaillès, his preference would be for the latter (Georges CANGUILHEM, *Œuvres complètes*, t. IV, pp. 269-270: "Hegelianism? I don't know. Spinozism? I would be more inclined to agree..."; the same position is taken by DESANTI, *Œuvres*, p. 6 ("he who wanted to be a Spinozist") and p. 8 ("he wasn't thinking of Hegel, whom he distrusted..."). Canguilhem's 1967 text quotes G.-G. GRANGER, who had entitled his first tribute "Jean Cavaillès ou la montée vers Spinoza" (*Études philosophiques*, nouvelle série, n^{o} 2, juillet-décembre 1947, pp. 271-279).

E. SCHWARTZ, in "Jean Cavaillès (1903-1944)" (*L'Auvergne en philosophie. 1. Portraits et itinéraires, Revue d'Auvergne*, no 580-581, 2006, pp. 309-310) and "Histoire des mathématiques et histoire de la philosophie chez Jules Vuillemin" (in *Philosophie des mathématiques et théorie de la connaissance. L'œuvre de Jules Vuillemin*, Paris, Blanchard, 2005), concludes rather in the opposite direction, in favour of a Hegelian influence. H. BENIS SINACEUR takes stock of this question in chapter VIII of her *Cavaillès*, op. cit. note 3. Among the other signs of the mark left by Spinozism on Cavaillès' thought, we might mention: singularity (see note 38 below); thought as action, the power of doing understood as autonomous, i.e. following its own law, in the production of ideas; the direct application of the order of intelligible essences to becoming and history.

in nothingness, being is necessary being'[20]". Considered in relation to the Brunschvicgian theorisation, a version of which we shall discuss below in the very different spirit of Georges Canguilhem, we might say that, like Spinoza's, Cavaillès's conception goes to the extremes in the path of necessity, or, more precisely perhaps, tends to eliminate, between the two poles of necessity and contingency, that of the merely possible[21]. This could be the essential significance of Cavaillès' reference to Spinoza.

It seems that it was in Dedekind that our author found the perfect example of what he meant by necessity. In *Méthode axiomatique et formalisme*, our author quotes from his habilitation speech[22] and

[20] *La Modalité du jugement*, op. cit. pp. 57-58.

[21] For the place given to contingency, most often linked by Cavaillès to unpredictability, see particularly the introduction to the *Remarques sur la formation de la théorie abstraite des ensembles*: the recognition of the role of "psychological and social chains of events," the observation of "unexpected shifts in mathematical development, its ironic abandonment of the paths that systematic forecasting had opened up before it..." (*Œuvres*, pp. 226-227). (*Œuvres*, pp. 226-227) and, in *La Pensée mathématique* (*Œuvres*, pp. 600-601 remarks on "picturesque contingency," set theory that has been "built on radically unpredictable inventions," and mathematical becoming that develops "like true becoming, that is, unpredictably." The main characteristic of unpredictability, which tends to take on the function of a historical category, seems to consist, for Cavaillès, in the impossibility for a content of thought to be deduced from a prior analysis of concepts, which brings it closer to the intuitive in the Kantian sense, one of his major philosophical references.

[22] DEDEKIND delivered his "habilitation lecture" in 1854 in Göttingen, in the presence of Gauss. It was not published until 1932, in the collection of his mathematical works (*Gesammelte mathematische Werke*, Braunschweig, 1930-1932, vol. III, pp. 428-438); P. DUGAC (*Richard Dedekind et les fondements des mathématiques*, Paris, 1976, p. 17) notes that Dedekind began his 1852 doctoral thesis (*On the Elements of Eulerian Integrals, Ges. Werke*, vol. I, pp. 1-26) with general considerations on the reasons for introducing new concepts in mathematics, and emphasized the positive role played by the difficulties encountered in the practice of "indirect" operations, whether elementary (subtraction, division, root extraction) or functional (integration as the inverse of differentiation), in broadening definitions and introducing new concepts. Cavaillès is therefore right to quote this "habilitation speech" in *Axiomatic Method and Formalism* (chap. I, 1, b; *Œuvres*, p. 61) for the "law of progress" in mathematics and its immanent necessity, illustrating it in particular by the extension of integers to real numbers in the theorem of addition of exponents in exponentiation. He quotes Dedekind again in his presentation to the French Philosophical Society (*La Pensée mathématique*, *Œuvres*, p. 602) and in *Réflexions sur le fondement des mathématiques* (*Œuvres*, p. 579) as constituting, with Hilbert, the direct source of his characterization of those processes of mathematicians that he subsequently calls "thematization" and "idealization." He returned to this theme, taking them in reverse order and presenting them in greater detail in his posthumous work (*Œuvres*, pp. 509-515, with the example of the Lebesgue integral added on p. 514 to that of irrational exponents).

uses the example of non-integer exponentiation: "The extensions of definitions leave no room for arbitrariness, but follow with absolute necessity from the primitive definitions if we apply the principle that the laws that follow from them are characteristic of the concepts they introduce and have universal validity". Cavaillès says that there is a "necessary engendering of new concepts", and this according to a double necessity. He recognises that this term can hardly be specified in any other way than by recognising certain requirements, requirements which are internal to the problems and the notions. The operation posed requires an enlargement of the field of objects; in this new field, relations require a new definition. It is therefore essentially a question of operative autonomy, that of the acts of the mathematician who carries out operations. In Dedekind's presentation of mathematical progress as a necessary, because required, introduction of new concepts, Cavaillès instead sees the "internal dynamism of autonomous mathematics": the independence of mathematics and its becoming, against any attempt at reduction.

Dedekind's hostility to any form of reductionism (particularly arithmetical reductionism) found expression in the vocabulary of "creation", a term that Cavaillès himself would use in his secondary thesis on Cantor's theory[(23)], without there being any apparent conflict with the necessity of the process: how can the latter be understood?

In fact, Cavaillès does not seek to absorb the entire development into a scheme of autonomy-necessity, and he knows perfectly well how to make room for contingency, so to speak, at the margins of such a scheme. If, in the *Remarks on the Formation of the Abstract Theory of Sets*, we refer to the description of the emergence of the latter, we see that, for him, in an authentic history, necessity is not absolute. The introductory remarks are certainly directed against a practice of history that would overemphasise the role of contingency, but they do not deny it. Cavaillès is not saying that the history of mathematics is not linked to what it conveys — this link is twofold, with problems and methods, and it is purely *a parte post* —

[(23)] He had already seen Brunschvicg use the term, noting, for example, that decisive discoveries in science give rise to "unforeseeable" types of relationship: "creation, in the strongest and most positive sense of the word, therefore characterises the order of intelligence : and thus the history of scientific thought is a real history, whose curve cannot be described or extended until the events that determine its course have actually occurred" (*Le Progrès de la conscience dans la philosophie occidentale*, Paris, Alcan, 1927, t. II, p. 705).

but that it is, of all histories, the least so. He does not deny that factors such as arbitrariness, the individual or the style of an environment have any value as historical explanations, but he claims that they are not enough to explain. He even acknowledges the difficulty of distinguishing between psychological or social sequences and mathematical necessity, or of answering the question: is there a necessary apparition, an autonomous structure, or historical fantasy, plurality embedded in a contingent system? The point is to maintain the presence of an irreducible core of objectivity in knowledge, of necessity in the historical development of mathematics.

Cavaillès, as we have seen, did not content himself with the task of revising, in light of the latest developments in logic and mathematical analysis, the conception that one could have of mathematical knowledge and the history of mathematics. He had a plan to use these foundations to help build a theory of knowledge, initially called the "theory of reason" and later the "philosophy of the concept". It seems to us that, here again, we can better understand the intention and the stakes if we accept to go back to Brunschvicz's interpretation of Kant. It seems to us that, here again, we will better understand the intention and the stakes if we accept to go back to the Brunschvic interpretation of Kant.

For Léon Brunschvicg, the Kantian theory of knowledge as a whole results from the effort to take account of mathematics as knowledge. Kant started from the clear-cut Humean opposition between truths of reason and truths of fact, from their radical dissociation: on the one hand, the logical use of analytical reason, and on the other, experience, which is irreducible to it. But the discovery of mathematical knowledge as both rational and synthetic bridges the gap between understanding and reality, giving rise to the possibility of certainty in science. This cannot be without consequences for understanding and its necessity, whose very concept must be modified. A new type of necessary judgement is required, which sets mathematical necessity against the analytic, to which Leibniz wanted to reduce it. This new type of requirement calls for a new legal status: this will be the synthetic *a priori* judgment. But, at the same time, Kant conceives of this mathematical judgement in a more traditional way than Descartes and Leibniz. While, in Descartes and Leibniz, the chain of mathematical truths is often irreducible to syllogism, Kant continues to conceive of this chain in accordance with the simple laws of formal logic and to base

their apodictic nature on the principle of contradiction. His solution consists in allowing analytical necessity to extend *a priori* to a real domain, insofar as a content of reality can be offered to it *a priori*, thanks to pure intuitions. In this way, we obtain the principles, propositions from which we cannot escape, but between which the connection remains exclusively logical[24]. Such a solution raises major difficulties, which the "theory of reason", in *Méthode axiomatique et formalisme*, or the "philosophy of the concept", in the posthumous work, could not fail to take into account.

These Kantian analyses, if we take them together with Brunschvicg's criticism of them, provide the origin and the framework, not only of the close discussion to which Cavaillès subjected Kantian doctrine on several occasions, but also, and above all, of what Cavaillès's conceptions of mathematics and the relationship between mathematics and logic were to be. In any case, they help to clarify them. This will be shown in three points.

The first concerns the importance Cavaillès attached to the Hilbertian doctrine of mathematics.

In his developments on Hilbert, Cavaillès does not fail to mention Hilbert's quotations from Kant. For example, in *Méthode axiomatique et formalisme*, chap. III, in the *Neubegründung der Mathematik* of 1922: "Already Kant showed that mathematics has a secure substance of its own, independent of any logic, and therefore can never be founded on logic alone, hence the failure of Frege and Dedekind. On the contrary, the condition for the application of logical reasoning is the presence of a given in the representation, certain concrete extra-logical objects that intuitively lie there as an immediate experience prior to any thought." Cavaillès interprets the reference to Kant as the search for a philosophical authority to support the idea of the autonomy of mathematics in relation to logic:

[24] See *Critique de la raison pure*, 2e édition, 1787, Introduction, § V : "As it was found that the reasonings of mathematics all proceeded according to the principle of contradiction (as required by the nature of all apodictic certainty), one became convinced that the principles of mathematics must also be conceived according to the principle of contradiction, in which one was mistaken because, although the principle of contradiction can make us accept a synthetic proposition, this can only ever be insofar as one presupposes another synthetic proposition from which it can be derived, but never in itself." That is why, when he wants to establish the necessary system of concepts through which the original understanding thinks intuitions, Kant naturally turns to the logical system of forms of judgment, which originally expresses in our minds "the nature of all apodictic certainty." And he believes himself entitled to extrapolate the necessity of transcendental understanding, which is synthetic, from that of logical understanding, which is analytic.

"mathematics is more than logic, insofar as it is effective thought, and all effective thought presupposes the application of abstract thought to an intuition". On the one hand, mathematics is an authentic mode of knowledge, which Kant recognised philosophically by assigning to it the field of *a priori* synthetic judgements; on the other hand, it is something other than logic, whose proper domain is that of the chaining or linking together of propositions, to the exclusion of the determination of propositions themselves, for which pure *a priori* intuition must be brought into play[(25)]. This is what Hilbertian analysis recognises, while differing from Kantian conclusions and giving them a different scope. As Hilbert notes in his *Mathematische Probleme* of 1900, quoted by Cavaillès, "Who does not use the drawing of embedded segments to demonstrate rigorously a complicated theorem on the continuity of functions, or the existence of accumulation points? Who could do without the figure of the triangle, the circle with its centre, the cross of the axes of coordinates?... Arithmetic signs are written figures, geometric figures are drawn figures, and it would be as impossible for a mathematician to do without them as it would be to ignore parentheses when writing". "The very essence of mathematics", comments Cavaillès, "is the regulated play of symbols". The concrete operation of combination determines, and this in law, "an irreducible region of intuitive reasoning", without which mathematics would not exist. In true mathematical work, the important thing is in the intuitive chains of reasoning. First and foremost, as Kant understood it, it is fruitfulness of knowledge that is guaranteed by recourse to intuition. But progress in knowledge does not take place passively, as a result of the unification of the diverse intuitively given to abstract thought. The mathematical symbol, or sign, is nothing without its rules of use, which are intellectual, and whose application results in a movement in the sensible, the true matter of creation.

"It is in intuition that the free act appears," and intuition is still an activity. This is one of the sources of what was to be a constant theme for Cavaillès.

[(25)] *Méthode axiomatique et formalisme*, chap. III, *Œuvres*, pp. 99-102. The only difference with Kant is that, for Hilbert, there is no such thing as pure logical thought, logic being only a constituent, not isolable, of any truly functioning thought. The role of the intellectual or the logical, Cavaillès shows, is as limited as possible in Hilbert's case: it is simply a matter of fixing the results obtained or the conventions adopted.

The second point is just as fundamental. In a key passage of his posthumous work, Cavaillès attempts to recapture the "constitutive properties of thought", which he calls "the paradigm" and "the thematic", on the basis of a particularly in-depth analysis of the process of demonstration. Linking rational progression to Kantian synthesis, he is keen to point out that "the synthesis that Kant discerns in thought does not require any provided or different diversity, other than itself, multiplicity through its moments and its progress..." The passage is directly evocative of the way in which, as we emphasised above in recalling Léon Brunschvicg's objections to the deduction of categories, Kant describes the synthetic power of the category when he seeks to grasp it absolutely, abstracted from all experience. The unification of terms is determined, not by the elimination or neglect of what may be different in them, thanks to the identity of a common character, but as the unity of the different as such. This generalisation of Kant's absolute categorical synthesis is itself, it should be noted, the result of an ontological conception of the relation for which Cavaillès seeks support in Plato's *Parmenides*: "The being [of the relation] is what it adds to its origin, hence is other than the necessity that makes it one, hence affirms in spite of this necessity an independence that translates into relative indifference, a source of plurality". And Cavaillès immediately links, in a very Brunschvicgian movement, the categorical unification thus understood to history or to what he will eventually call "progress", since he assigns as the true object of this unification, no longer the intuitive diversity of representation, left to the "spatial imagination", but "the very sequence of acts, as each of them, forgetting itself and realizing itself at the same time in a meaning, can only establish its own being as an element of a whole recognized as plurality and, immediately, as the starting point for new acts", making synthesis "coextensive with the engendering of the synthesised"[(26)]. The analysis will be taken up on the last page.

The final point concerns the doctrine of concept itself.

The difficulty of Kant's solution to the problem posed by mathematical knowledge, namely the extension of analytic necessity to the intuitive real, is linked to the inadequacy of his theory of the concept. Kant's main argument against Leibniz in mathematics, which can be found in several passages of the *Critique*[(27)], is as follows: it

[(26)] *Sur la logique et la théorie de la Science*, *Œuvres*, p. 510.

[(27)] *Critique of pure reason*, 2e édition, 1787, Preface p. 8, Transcendental Aesthetic: General Remarks, or Introduction V, or Transcendental Doctrine of Method: The

is impossible to derive from a simple concept propositions that go beyond it, as happens constantly in geometry. The series of geometrical theorems marks an increase in our knowledge, and not just an increase in the clarity of a knowledge already virtually within us, as Leibniz wanted. To explain this real increase, we need a synthesis, something that is added to the concept, that cannot be extracted from it, and that we must therefore look for elsewhere, otherwise our progress in knowledge would lack reality. The basic difference between Kant and Leibniz, then, is their idea of the nature of the concept. For Kant, the concept, considered in itself, is no more than an empty framework, capable of a purely nominal definition. For Leibniz, a true mathematician, the concept is a positive essence, a purely intelligible reality, which includes not only a nominal definition, but a real definition, from which we can develop properties ad infinitum.

From this point of view, Kant's doctrine of the concept, impoverished intelligible, could not satisfy Cavaillès, who here follows entirely in the tradition of classical rationalism derived from Descartes.: that of the understanding as the power to think, the positive power to generate concepts that are full concepts, saturated with intelligibility, adequate to the knowledge of things. The essence of Cartesian "simple nature" had drawn from its algebraic type of abstraction a character of singular universality that made it capable of founding a science of pure understanding. And this is even more true, if possible, of Spinoza, who, as we have seen, marked in a certain way, for Léon Brunschvicg, who never hid his Spinozist affinities, a kind of fulfillment of the Cartesian moment. It is understandable that Brunschvicg should feel entitled to indulge in anachronism for a moment by proposing to see in the Cartesian judgement of relation the uncovering of the true nature of *a priori* synthetic judgement, freed from the Aristotelian prejudice in favour of concepts and categories[(28)]. For Descartes, the singularity apprehended by the understanding makes it possible to grasp, in the simplicity of a single, singular formula, the law of a multitude of other reasons, themselves singular and of a more restricted domain.

Discipline of Pure Reason in the sphere of Dogmatism. Take, for example, Kant remarks, the proposition: two straight lines cannot enclose a space, nor therefore form any figure, whereas, on the contrary, with three straight lines one can form a figure, and if one tries to derive it from the concepts of a straight line and the number two (or three), one will soon realize that all one's efforts will be in vain...

[(28)] "Sommes-nous spinozistes", Écrits philosophiques, vol. I, p. 159.

The significance of singularity, the absence of extension as defined by logicians, is here to mark the elimination of the conception of abstraction as an empirical generality whose truth would only come from the sensible things from which it is derived. The Spinozist "singular essence" is incommensurable with a generic concept[29].

In the oft-quoted programmatic development that closes the posthumous work, Cavaillès asserts that a theory of science can only come from a "philosophy of the concept", the only one capable of enabling us to think the "generative necessity", which is that of a "dialectic". Progress, he points out, takes place "between singular essences", and "what comes after is more than what came before, not because it contains it or even prolongs it, but because it necessarily emerges from it, and bears in its content the mark, each time singular, of its superiority". So, in what may be considered his ultimate thought on history, we find the domination of necessity. It is expressed in a philosophical vocabulary that borrows both from Spinoza ("necessity", "singular essence") and, a more recent reference, from Hegel ("dialectic", "moments of consciousness") — although it is not certain that these notions can be made to correspond in content to the doctrines they evoke[30]. Interpreters of these somewhat enigmatic passages have naturally emphasised one or the other, depending on their philosophical sensibilities. It is of course impossible, and probably rather futile, to decide between the two, assuming that the references are incompatible, which they certainly were not in Cavaillès's mind. In an

[29] Along with necessity, singularity is undoubtedly the element of Spinozist metaphysics that was most likely to appeal to Cavaillès. For Spinoza, all reality is singular, and authentic truths are singular truths. If "intuitive science" stands above abstract thought, it is because it is ideally knowledge of reality in its singularity, which has no common measure with merely general or even universal knowledge. This superiority of singular knowledge of course ultimately derives from the uniqueness of substance, from the thought of God as a Being that is not only infinite but singular. It should be noted that singularity, the character of a representation that can only be given by a single object, is, along with infinity (infinite continuity), one of the two properties chosen by Kant to characterise space as pure intuition: see *Critique of Pure Reason*, Transcendental Aesthetic: Of Space, Metaphysical Exposition of the Conception of Space.

[30] This is particularly true of the term "dialectic", which Cavaillès sometimes applies adjectively to activity, whether of the mind or of consciousness, especially in his early writings. For him, it certainly seems to be a question above all of setting a limit to the domination that one or the other could claim over their contents, and of which the omnipresence of consciousness in Husserl provides the most complete form, albeit ultimately demonstrably illusory (*Sur la logique..., Œuvres*, pp. 552-558). For the relation to Spinoza, see notes 18 and 19.

earlier work[31], we thought it possible to shed some light on this development of singular essences by looking at a particularly noteworthy episode in the history of mathematics, one that was far from unknown to our author, since it concerns the concepts of measure and integral. Think of the concepts of Cauchy integral, Riemann integral, Borel measure, Lebesgue measure and integral, Stieltjes measure and integral, then Stieltjes-Lebesgue measure, Radon measure, measure and integral in the Daniell sense. We find their mathematical intelligibility in their definition and their relationship to theory, and their status as "singular essences" in the fact that their characteristic properties manifest, at each of their historical appearances, a superiority that can be expressed in their own operative terms. Whether it concerns, in the functional domain, the properties required of the function to be integrated (its continuity, monotonicity, etc.), or, in the domain of definition, the nature of the domain over which it is integrated (finite or infinite, real or complex, including or excluding points of discontinuity, etc.), each achievement of generality translates into an increase in operational possibilities, for properties that are specific in each case.

Thus, the Riemann integral makes it possible to escape the (global) constraint of continuity on the functions for which Cauchy had defined his integration procedure, and thus to require only integrability in the Riemann sense for them, a more general condition than continuity. Thus, compared with Jordan's measure, which is limited to the finite, Borel's measure reaches countable infinity, a singular sign of progress that will make possible Lebesgue's definition of its integral. Thus, compared with the Riemann integral, the Lebesgue integral allows, among other operational advantages, the passage to the limit on increasing sequences of functions, which makes it possible to reach completion in the spaces of functions that can be defined from it. As for the "Stieltjes measure", its distinctive feature is that it generalises notions of physical origin linked to the measurement of quantities, that of pointwise measurement, defined in a direct manner, independent of any prior definition of the quantities, and then that of weighted sum. The Stieltjes integral, for example that of a function defined on a real segment divided into intervals, can be considered as the weighted

[31] *Constitution de la théorie moderne de l'intégration*, Paris, Vrin, 1992; Cavaillès analyses the work of Borel and Lebesgue in the introduction to *Méthode axiomatique et formalisme* (*Œuvres*, pp. 14-29), and cites Lebesgue's integral as an example of what he calls 'thematisation' (*ibidem*, p. 515).

sum of this function for a measurement defined as taking as its value, on each interval, the difference of the values taken by the function at the limits of the intervals[32]. The operational advantage of the concept will become apparent later, with the representation theorems it makes possible, especially after Riesz showed that any linear operation can be expressed by the Stieltjes integral of a function subject to a certain well-defined property (being of "bounded variation"). The integral becomes an "operator", i.e. a means of obtaining functions from other functions, a fundamental point of view for functional analysis. The combination of Stieltjes' integral with Lebesgue's integral, its formulation in terms of linear form, which opens up new fields of operation, leads to modern definitions.

It remains to specify the differences that separate the conceptions of mathematical history that were, in our opinion, those of Cavaillès, from those, more general, of Brunschvicg concerning the history of scientific knowledge. They may indeed retain their full meaning, if we are willing to accept the position in which we hold the main question they cover, that of the modality of historical judgement.

For Cavaillès, fidelity in that which is mathematical in a thought requires establishing the objectivity and necessity of its becoming. For the first: "there is an objectivity, mathematically based, to mathematical becoming"; "even if we conceived of mathematics as a system in itself, the twists and turns of the process of revelation would be related to the structure of the parts revealed."[33]. For the second: "mathematician, i.e. revealer of necessities"; "the image of the gesture should not be misleading: however gratuitous the invention of a method may seem, the development of mathematics takes place according to a necessary rhythm"; "autonomy, therefore necessity"[34]. The problem stems from the fact that objectivity and necessity are, in themselves, negations of history.

[32] More precisely, if we are dealing with a monotone and non-decreasing function f, assumed, to fix ideas, to be "continuous on the left", on a real segment (a, b) divided into intervals (a, b), the integral will be the weighted sum of the function f for the measure defined on the half-open intervals $]a, b]$ by the relation : $m(]a, b]) = f(b+) - f(a-)$, the latter expressions designating the limits, on the right and left respectively, of the function f at points a and b. We refer to this as the "Stieltjes measure" generated by the function f.

[33] *Remarques sur la formation de la théorie abstraite des esembles*, p. 28, *Œuvres*, p. 226.

[34] Respectively: *Œuvres*, p. 226 (*Remarques...*), p. 664 (*Mathématiques et formalisme*), p. 601 (*La Pensée mathématique*).

Consequently, the history of mathematics can only be a very particular history, to be taken in a sense that cannot conform to the ordinary meaning of the word history. This is why Cavaillès sometimes says of this history that "it is not a history", or that "there is nothing so unhistorical — in the sense of becoming opaque, graspable only in an artistic intuition — as mathematical history"[(35)]. Thus history, when it is the study of a mathematical object, is forced, so to speak, by the nature of the latter, to dissociate itself from "that of which it is a vehicle", its contingent matter. This disengagement from the contingent, which amounts to the revelation of a necessity, results in the elimination of history in the empirical sense. Once the history has been accomplished, we realize that contingency, in its arbitrary aspect, was in fact only an appearance. The necessity was present; it becomes apparent in retrospect, upon analysis, in the revealed sequence of concepts: there was a constraint imposed by the problems and methods, a reciprocal conditioning of notions and theories, an interdependence of the parts among themselves and with the whole. Necessity is only *a posteriori*, the end of a work of elucidation and reconstruction, which is precisely that of the true historian. Hence the refusal of any rational teleology, and the legitimate role of the unforeseeable, of adventure, in the scientific enterprise. Cavaillès underlined this point in relation to the abstract theory of sets. Becoming does not follow a plan, it is not designed in advance. As stated in his posthumous writings, one could say that it presupposes "movement as irreducible, and therefore the risk of departing from the Self, of embarking on an adventure toward the Other, who is both already there and not yet there, who may disappoint despite our expectations, who moves at his own pace[(36)]".

We would then find, behind all the nuances of an inimitably scrupulous thought, the lesson of the modality of judgement. Today, we might ask ourselves, even if we limit ourselves to the

[(35)] *Méthode axiomatique et formalisme*, p. 176; *Œuvres*, p. 184.

[(36)] On the dialectic of prediction according to Husserl, *Sur la logique et la théorie de la science*, 2e édition, Paris, 1960, p. 68; *Œuvres*, p. 550. The rejection of all rational teleology is one of the negative elements in the debate on Cavaillès' (final) proximity to Hegelianism. It is, in any case, a point of divergence with Bachelard, who takes it somewhat to his own advantage, it seems, in his study *L'Œuvre de Jean Cavaillès* (first published in Gabrielle FERRIÈRES, *Jean Cavaillès, philosophe et combattant* (*1903-1904*), Paris, Puf, 1950, republished in a collection of articles entitled *L'Engagement rationaliste*, Paris, Puf, 1972, pp. 178-190). Referring to Dedekind, Bachelard states that Cavaillès discovered in Dedekind "a regular origin", "the foundation of a reconstructed science" (p. 182).

point of view of the practice of history, whether we can go so far in the direction of necessity — we would willingly say: whether it is *reasonable* to go so far in the direction of necessity. The history of mathematics cannot be dissociated from the history of other sciences, those we sometimes call the sciences of reality, whose history does not seem to obey such an exclusive form of modality. In accordance with Léon Brunschvicg's analysis, we find the original presence of possibility, ineliminable by necessity. As Georges Canguilhem, who was also Brunschvicg's pupil, has luminously shown, in the effective progress of science in general, a Bergsonian rule of retroactivity of truth applies: to judge a proposition as true is to confer upon it a retroactivity of validity that results precisely in its being removed from history. Once completed, verification appears to be the almost mechanical effect of an impersonal necessity that manifests itself irresistibly. But the fact that it is inscribed in time brings us back to the idea that scientific truth first existed only as a possibility. The object of knowledge first presents itself as a possibility, and it is its subsequent validation as necessary that produces it as a real[37]. But in this historical production of true knowledge, the genesis of the possible matters as much as the validation of the necessary, and the fragility, or precariousness, of the first moment, the possible — freely, hypothetically, posited — does not deprive it of a value that would confer on the second, the necessary, its provisional solidity, or stability. Can it be otherwise in the history of mathematics? It would be necessary to recognise that mathematics is in an exceptional situation that is rather difficult to justify, and it would be better to admit that in the history of mathematics, as in that of any other science, there is no "final scientific judgement".

"Behind the words 'we know'", says Georges Canguilhem, "are the words 'we haven't always known'." What we are looking for, we don't know how we will find it, otherwise it would already be there, or already in sight. And that is why history cannot be written as a teleology, as if it were heading from its beginnings towards its end, the objectivity of the final result, but as an adventure, which

[37] In this sense, we could modify the Kantian formula, which gives the necessary (scientific knowledge) as a synthesis of the possible (the categorical) and the real (the empirical sensible given). In its historical process, the real of knowledge is generated from the possible and then from the necessary, and we could say, this time reversing the Leibnizian formula: if the possible is necessary, it exists.

it authentically is[38]. We must recognise that Cavaillès's thinking, while accepting and integrating this fact into his analysis, as we have seen, did not feel it necessary to concede any more to an initial freedom of choice, and to introduce into the movement of history the initial unfolding of its possibilities.

Take, for example, the theory of ideals, a fundamental theory for understanding the conceptions of Hilbert and Dedekind, which themselves directly inspired Cavaillès's views on generalisation and mathematical progress[39]. Today, we know from the work of M. Eichler, H. M. Edwards and, most importantly, A. Weil[40], that Kronecker's approach to the concept of the ideal was in no way mathematically inferior to that of Dedekind. H. Weyl himself expressed his preference for Kronecker's theory, a modernised version of which he developed in his book on the algebraic theory of numbers[41]. We know that Kronecker's theory of divisors not only does exactly what Dedekind's theory of ideals did: it generalises

[38] See *La Connaissance de la vie*, Paris, Vrin, 1965, p. 47; *La Formation du concept de réflexe*, Paris, Puf, 1955, p. 156. In the first text, Canguilhem uses the formula: "L'histoire, c'est-à-dire selon nous, le sens de la possibilité". In his 1989 tribute (Une vie une œuvre...), he quotes the following sentence about his friend, taken from *La Pensée mathématique* (*Œuvres*, p. 594): "the mathematician is embarked on an adventure that he can only stop arbitrarily and each moment of which provides him with a radical novelty", but he immediately recalls that, for Cavaillès, necessity appears as soon as one thinks in the long term: "Necessity. The thought, work and conduct of Cavaillès are a recognition of necessity" (*Œuvres*, p. 684).

[39] The two major texts from which Cavaillès himself says he drew his presentation of mathematical conceptualisation (*Méthode axiomatique et formalisme*, *Œuvres*, pp. 61-63, 104-105) are: 1) Dedekind's 1854 habilitation speech already cited (note 25) on the necessity of the introduction-creation of new objects in the course of mathematical history, 2) Hilbert's memoir on infinity (text of a lecture delivered in 1925, "Über das Unendliche", *Mathematische Annalen*, t. 95, pp. 161-190; trans. by André Weil, "Sur l'infini", 1926, *Acta Mathematica*, 48, pp. 91-122), in which he mentions the idea of using the Kummerian method of ideals to make sense of the introduction of transfinite elements into mathematical systems. It is as a typical example of mathematical generalisation, and an instrument for characterising "idealisation" or "paradigm", that the method of adding ideals plays a crucial role. We take up here the development of an earlier work, in which we tried to show how, in relation to the historical distribution operated in Cavaillès's first works, the direction taken by mathematics forced us to reconsider in depth the case of KRONECKER's arithmetistic intuitionism ("Après Jean Cavaillès, l'histoire des mathématiques", Philosophia Scientiae, 1998, vol. 3, cahier 1, pp. 113-137).

[40] M. EICHLER, *Einführung in die Theorie der algebraischen Zahlen und Funktionen*, Basel, 1963, English translation, *Introduction to Algebraic Theory of numbers and functions*, New York, 1966 ; H. M. EDWARDS, *Advanced Calculus*, Boston, 1969 ; A. WEIL, *Number theory and algebraic geometry*, Princeton, 1950.

[41] H. WEYL, *Algebraic Theory of numbers*, Princeton, 1940.

Kummer's theory of ideal factorisation from a field of cyclotomic numbers to a field of algebraic numbers, but it also avoids two major defects of Dedekind's theory, namely the lack of intrinsic character and the absence of an explicit construction of divisors. It therefore seems rather difficult to maintain that it was necessary here, as some of Dedekind's formulations might have led us to believe, to involve ideals in the sense of sets.

The historical judgement seems to have been obscured by an abusive reduction of the issue to that of comparing Kronecker's doctrinal conceptions, often caricatured, with those of Dedekind or his old adversary Cantor. "While", says A. Weil, "every line of Dedekind's XIth supplement, in its three successive and increasingly 'pure' versions, has been scrutinized and analyzed, axiomatized and generalized, Kronecker's famous Grundzüge... have been almost forgotten, or are perceived as presenting an inferior and less pure method for obtaining the same results[(42)] [...]" And A. Weil pointed out how Kronecker's goal was much broader than dealing with the fundamental problems of ideal theory, which was Dedekind's main subject. For him, it was a question of describing and inaugurating a new branch of mathematics, which would have embraced both number theory and algebraic geometry, but as separate branches: a truly grandiose conception, which he alone did not have the means to bring to fruition, but on which the developments that these branches have undergone in recent times allow us to have a view that is undoubtedly more accurate than the summary condemnations of Kronecker's "arithmetist" conceptions. The final and unqualified success of the Cantorian theory of infinite sets, and its essential consequence, the recognition of the complete infinite totality as a fundamentally legitimate object of mathematics, which had become a doctrine condemned by Kronecker, got the better of his project. What Kronecker considered to be the greatest virtue of his own work — constructing his definitions, giving demonstrations of existence in algebraic, finite and, to put it bluntly, algorithmic terms — found itself covered over for a long time.

For a long time, but not for good. History was preparing its revenge for Kronecker, in the form of the advent of calculators, and

[(42)] *Number-theory and algebraic geometry*, 1950; *Œuvres scientifiques*, vol. 2, pp. 442-452. The work by Kronecker to which A. Weil refers is of course the great treatise (meditated and matured over some twenty years) that he published in Berlin in 1882: *Grundzüge einer arithmetischen Theorie der algebraischen Grössen*, Werke, vol. 2, pp. 239-387.

the success of a new school of thought, which, to put it succinctly, can be called algorithmic thinking. The ability to test hypotheses and calculate data faster and more easily than ever before has changed not only the way we deal with problems, but also the way we think about them. From this it can be seen that there were other possibilities for development than the path of abstract set-theory that almost all mathematicians were to follow in the footsteps of Cantor and Dedekind. It had been possible, and this is precisely what Kronecker's work attests to, to think of mathematics in terms that differed from those of Cantor and Dedekind, that is to say, even if with requirements that differed from those of today, in algorithmic terms. Rereading Kronecker today, it appears that there is no a priori mathematical privilege to a set-theoretic expression of concepts, even those recognized as fundamental. In fact, neither Dirichlet nor Gauss, whose style also deserves to be described as conceptual, and who also had a firm conception of generalization, would undoubtedly have undertaken to formulate fundamental mathematical concepts in terms of sets, any more than they would have accepted the legitimacy of using infinity as a completed entity. Thus, what *Transfini et continu* calls Kronecker's "fine arithmetism", and which he compares, without any depreciatory nuance, to the similar attempts of Weierstrass (arithmetisation of the infinitesimal calculus), Hilbert (finitism) and Dedekind himself (whose motto he quotes: "man always arithmetises"), undoubtedly did not receive, on the part of Cavaillès, the attention it deserved[(43)]. One might be tempted to say that this very fact testifies to the irreducible freedom that remains in historical judgement, even when it comes to mathematics, since in real history there is no necessary accomplishment that has not first been the unfolding of possibilities.

For Cavaillès, the work of reflection, which Brunschvicg saw as the hallmark of philosophy, primarily concerns mathematics itself. What the "profound work of the Borel-Lebesgue school" should lead us to, he explains at the beginning of *Méthode axiomatique et formalisme*, is a "critical reflection on the very essence of mathematical work", a "systematic revision" and a "regression that leads us to dig beyond mathematics itself, into the common soil of all rational activities"[(44)].

[(43)] In the references to the two theses and the disclaimer to the Cantor-Dedekind correspondence (the latter transcribing Cantor's judgements), the judgement on Kronecker is directly inspired by Dedekind's, and his contribution is reduced to the arithmetical reduction of the analysis.

[(44)] *Œuvres*, p. 29.

To take mathematics as a quasi-experimental work, operating on singular contents, and then as dealing with a "matter" in which it is affirmed that the subject of history, and not the individual subject or consciousness, consists, to put the "theory of reason" into perspective from there, was indeed to distance oneself from Brunschvic idealism, and even, in a way, to reverse its orientation. However, Cavaillès never ceased to think that this grasp of the essence of mathematical work could not be achieved without the use, even renewed, of the concepts or categories of philosophy. In this, he was not unfaithful to Brunschvicg's lesson, and it remains true that this use cannot really be understood, in his work, without the mediation of the interpretations that the master had given of Kant, Descartes or Spinoza.

Cavaillès was sensitive to historical contingency, openly echoing Dedekind's expression of his confidence in the capacity for free mathematical creation, and anxious to remain as close as possible to the actions or deeds, of the mathematician. He wanted to think of mathematics, and mathematical history, in terms of necessity. He did so with the same coherence, the same determination to see consequences through to the end, the same brilliance, in a word, that illuminated his actions as a member of the Resistance. But it is a fact of experience that once philosophical intervention, however powerful, has taken place in history, the course of history gradually but inevitably reveals its shortcomings. Without escaping the rule, the work of Cavaillès may well render us one last service, that of helping us to see more clearly the limits of a conception of history that would fall within the framework of a philosophy of necessity. The field of reflexive history, as he practised it for the disciplines of logic and mathematics, with his Brunschvicgian stamp, and in the time that tragic events have left him, can legitimately be judged today to be a little narrow. We might like to revisit his interpretation of historical necessity, to reconsider his exclusion of ontology when it is based on logical theory, and to broaden the scope of his thinking, even to the point of giving greater prominence to the events of empirical history and the suggestions of physics. The greatness of Jean Cavaillès lies in the fact that these shifts in the practice of history, which may be desirable today, only truly make sense to us if we claim his own, and thus accept his lesson.

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