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Mathematics and 'Depth': the example of number theory

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Mathematicians have often referred to certain results — usually statements, formulas or theorems, or even past or present problems — as 'profound'. But what exactly do they mean by this? They also sometimes speak of 'power' (especially for a theory) and, in another register, of 'beauty' or 'elegance' (especially for a proof). Such expressions are part of the language of comments that goes with any discourse on science. Is this just rhetoric? On the contrary, our hypothesis here is that such use is an indication of epistemological analysis. Assuming that mathematicians are not the least well placed to assess the value or significance of a mathematical result, we have to conclude that reflection on mathematics can only benefit from an elucidation of the meaning of these spontaneous comments. What is then their general meaning?

Of the more or less common terms mentioned above, 'depth' is undoubtedly the most interesting and problematic in terms of interpretation. Unlike beauty or elegance, it does not clearly belong to the aesthetic realm. The question of whether mathematics is a science or an art is an age-old one that has often been asked.(1) While it is difficult to answer, at least the meaning is clear. The same cannot be said for the question of depth. Moreover, depth does not belong to the class of terms that could be used to refer to internal features of mathematics, such as 'force' or 'power'. Indeed, it seems easier to give these terms a precise meaning — if not an epistemological status — simply because logic offers us strict conditions for defining their use.(2) There is something more hidden, and therefore more attractive, about 'depth'. One might be tempted to say that the issue here is clarifying the process of mathematical creation, but this can only be intrinsically different from the aesthetic process, since it is clear that the subject here is essentially theoretical. Can we gain a conceptual understanding of the conditions of profound results in mathematics?

(1) According to André Weil. Kronecker had put this question to Eisenstein when the latter defended his thesis (introduction to the edition of Eisenstein's works, *Œuvres scientifiques*, Paris, vol. III, p. 400). The same André Weil, in his 1936 letter to Simone Weil, ponders the possible comparison between the creation of a work of art and that of a mathematical theory: *ibid.,* vol. 1, pp. 254-255.

(2) A theory T_1 is said to be more powerful (representatively) than a theory T_2, either when it can be interpreted in T_2 by means of definitions, or when there is a representation (for example, a Gödelian or Tarskian representation) of T_1 in T_2. For example, the usual Cantorian theory of sets is more powerful than the theory obtained by deleting the infinity axiom in the same theory.

We can at least attempt to outline such an approach by examining a few historical examples.(3)

1. Although Jean Desanti's reflection on mathematics is not primarily concerned with making sense of the various categories of the mathematician's language, it has encountered the problem of depth. In *Les Idéalités mathématique*,(i) Desanti isolates three predicates at a certain point in his analysis in order to apply them to what he delimits as a 'domain of reactivation" in the 'reflexive field': power, richness and depth — the latter of which is said to be 'the most difficult to pin down'. The link with the problem of creation is recognised since it is asserted that an ideality can be more or less fertile depending on the region of the field through which it is approached; this region will determine whether it engenders a 'superficial' or 'profound' domain of reactivation. Desanti's conclusion illustrates the scale of what is at stake and the difficulty of the task: 'It is in this metastable depth that the regulated, yet non-mechanical, game of mathematical creation appears to unfold'.(4) The notion of *reactivation*, proposed at the end of *Les Idéalités...*, may enable us to establish a framework for an analysis properly articulated with history. As we know, one of Desanti's key epistemological theses is that an ideal object, and a mathematical ideal in particular, is never simply given or 'offered by its mere presence', but always 'through the mediation of the regulated system of designations that make it possible to handle it': 'Every object-ideality is grasped in the form of a field-ideality' (pp. VI–VII). However, reactivation is nothing other than mediation in action — the exercise of all mediations. Thus, the mathematical entity represented by $\sqrt{2}$, for instance, is silent in itself and only reveals its possible meaning through the mediation of a symbolic language. This language then frees up the field of its possible effectuations, such as the root of an algebraic equation, a Dedekind cut on the set of rational numbers or the limit of a Cauchy sequence of real numbers and so on. Such effective verifications of meaning occur in specific

(3) We will base ourselves here on an essay by G. Granger, *What is a Profound Result in Mathematics?* Philosophy of mathematics today, E. Agazzi & G. Darvas eds, Kluwer Ac. Publ. 1997, pp. 89-100.

(4) *Les Idealities...*, p. 281.

(i) *Les Idéalités mathématiques. Recherches épistémologiques sur le développement de la théorie des fonctions de variables réelles.* Éditions du Seuil (L'ordre philosophique), Paris, 1968, 2008. [Ed.]

contexts each time. They open up many different operative possibilities and represent many 'thematic' aims of the properties of the object $\sqrt{2}$. This last epithet broadly designates what concerns the properties of objects rather than objects themselves. Mediations are organised within a closed, circular system of indefinite *cross-references* (*renvois*) between these possibilities and objectives. There are always possible *pathways* (*voies de passages*) between them. The field is structured in depth, in layers (it is always shown in perspective, *cf.* p. 121 *ff.*): in short, it is temporalised. Without going into the details of the analyses of 'temporal horizon structures', we can say that reactivating an ideality involves changing the function of a region of the field. Initially defined as the goal of a reference (*renvoi*), it now appears as a source. Equality or equivalence between two expressions of integers, for example, is a formal relation which can refer to very different contexts depending on how we consider them in terms of the elaboration and complexity (richness and power) of their mathematical content. For instance, consider integers in the context of elementary arithmetic (i.e., addition). Now consider the same number in a more sophisticated mathematical context, such as an analytical expression of the 'factorial' function (i.e., a limit) or as an index or a topological invariant, for example in algebraic topology.

In order to understand the meaning of such equality, it is important to consider the mediations that give them meaning and then to return to the theoretical system of which the object-term of the equality is a part, i.e., in which the equality 'operates' (*fonctionne*). This allows us to return to the initial goal of the reference (*renvoi*): to reactivate the object by creating a connection between previously dissociated domains. This establishes a link with the problem of creation and opens up the field of history, as the paths of passage are unprecedented. In the middle of the nineteenth century, for example, Riemann could be said to have 'reactivated' Archimedes in the sense that his concept of the integral was defined as the limit of sums that corresponded directly to those considered by the Alexandrian mathematician.

Although the concept of history is present, Desanti's analysis does not focus on actual history. Regarding reactivation, Desanti notes that it is unclear whether Riemann was an assiduous reader of Archimedes. However, he argues that this in no way negates the fact that a whole region of the Archimedean field was within reach of reactivation for the idealities that Riemann produced. As we were warned at the outset, the philosopher must reflect, and it is the philosopher-epistemologist's role to ask how mathematics is produced, reconstruct the movement of knowledge production and decipher the network of meaningful connections that constitute this mathematical universe as a cultural phenomenon. Within the framework of such an analysis, the problem of the meaning to be given to depth in mathematics can only be the epistemological determination of the notion. At the same time, the field remains open to historical analysis. On the one hand, this analysis can be used for its own purposes; on the other hand, a more inductive approach can be taken, based on historical examples. We might even hypothesise that history and philosophy offer no more than an opportunity to verify that, however fine-tuned and precise an epistemological analysis may be, it leaves an ineliminable residue.

2. Let's start by looking at a few examples and trying to identify the characteristic thematic components of a depth. Broadly speaking, these components can be grouped into three themes: unpredictability, generality, and fruitfulness.

The first of these features can be explained by a more general fact: *the opacity of content*. As we know, Cavaillès strongly emphasised that mathematical truths were unpredictable. He even considered this to be a fundamental feature of their development, alongside necessity. According to him, it is in this sense that the history of mathematics is a true history: it does not develop according to a plan, but thwarts forecasts and reveals surprises instead. This explains his comments on abstract set theory, where he talks about 'these unexpected inflections of mathematical development' and its 'ironic abandonment of the paths that an attempt at prediction opened up before it'. This is the antithesis of Wittgenstein's views in the *Tractatus*, (6.22) where he states that if mathematics shows in equations what logic shows in tautologies, then it is understandable that there can never be any surprises in mathematics or logic (6.1251).

Let's consider the simple example in number theory of the method of proof known as 'infinite descent', which was first formulated by Fermat. We know that if a sequence of natural numbers is decreasing, it is necessarily stationary and cannot be strictly decreasing. This method shows that a given property or relationship is impossible for any number by proving that if it were true for a particular number, it would also be true for smaller numbers. This same argument then implies that it would be true for even smaller numbers and so on ad infinitum. However, this is impossible because any decreasing sequence of numbers must necessarily end. In short, if the hypothesis that a given positive integer has a certain property implies the existence of a smaller integer with the same property, then no integer has that property.[(5)] Some of Fermat's most beautiful results can be traced back to this idea, yet no one had ever suggested it before. It is a genuine mathematical creation, and in his letter to Carcavi of August 1659, which is like a mathematical will, he talks at length about this unforeseeable singularity:[(6)]

> "As the ordinary methods described in the books were insufficient to demonstrate such difficult propositions, I finally found a rather unusual way to achieve it. I call this method proof the 'infinite or indefinite descent'. Initially, I only used it to prove negative propositions, such as [...] that there are no right-angled triangles whose area is a square number.
>
> However, this method can also be applied to affirmative propositions, such as any number being square or composed of two, three, or four squares.
>
> I show that if a given number were not of this nature, then there would be a smaller number that would not be either. Then there would be a smaller number than that, and so on ad infinitum. From this, we can infer that all numbers are of this nature."

It was methods such as these that enabled him not only to prove his 'little' theorem: *if p is prime, and a an integer, then p divides $a^p - a$*, which is one of the most fundamental properties of integer arithmetic...and to study the numbers now known as Fermat numbers, of the form 2^{2^n+1} (which are not necessarily prime, although any

(5) It is therefore a kind of adaptation to integers of the refutation by the absurd. Naturally, as the text quoted below shows, inversion is possible, which demonstrates a given property by prohibiting the determination of a procedure whose repetition would lead to an infinite descent of numbers. Many examples of the use of this method can be found in Fermat himself: for example, he established the fundamental property of the division of natural numbers, or, in a margin of his copy of Diophantus's *Arithmetica*, the fact that the area of Pythagorean triangles (rectangular triangles whose sides have integer lengths) cannot be square.

(6) Quoted in J. Itard, *Arithmétique et théorie des nombres*, Paris, 1973, p. 41-45

prime number of the form $2^k + 1$ is of this form),[7] as well as to state propositions that had remained in the state of hypotheses, such as Fermat's last theorem.[8]

To make things a little more precise, we should start by saying that in mathematics, this opacity is that of content. We can also use this to justify Wittgenstein's remark, who was probably thinking at the time of formal mathematics that is similar to, and perhaps a simple development of, formal logic. Mathematics produces content and, we might add, singular content. If a whole aspect of profound results lies in the revelation of the unknown, this is because a *sui generis* reality first imposes itself as something unexpected. This is undoubtedly why Cavaillès often describes the mathematical process as singular and even speaks of a 'singular essence' in the final pages of his posthumous work. Granger also highlights this phenomenon when he points out that the transparency of the rules, based on the adequacy of the object and the operation, which leaves nothing to chance, as Wittgenstein describes it, is not maintained beyond elementary logic. Unlike in formal logic, where consistent and complete systems such as the calculus of propositions are used, the application of rules in mathematics does not exhaust the reality of the object. As soon as we enter the field of properly mathematical objects — even in the calculus of predicates — the functioning of the rules is no longer self-evident; the existence of the obstacles revealed is a sign of residual opacity.

This specific consistency of the object was manifested in mathematics very early on, as soon as rules and objects were established, i.e., as soon as calculation began. This occurred even before Greek

[7] A magnificent example of the singular and unpredictable nature of the properties of these numbers was provided by Gauss, who demonstrated — at the age of 18! — that, when they are prime, the circumference can be divided into n equal parts (or a regular polygon of n sides can be constructed) using a ruler and compass (conversely, the only regular polygons that can be constructed using a ruler and compass are those for which: $n = 2^k p_1 p_2 \cdots p_m$, where p-numbers are distinct Fermat prime numbers, and $k > 0$).

[8] Note that a profound mathematical fact is often associated with the formulation of conjectures about it, especially in the field of number theory and related fields. We know that there can be a long delay between such conjectures and their demonstration (see the example of Fermat's conjecture). It is as if the mathematician, finding himself initially unable to grasp the profound fact by means of the strict procedures of demonstration, was placed in the necessity of proceeding by intuitive apprehension of the result. Hence the value, and the role, of theoretical intuitions in the development of mathematics, too often underestimated in favour of the mere demonstration of results.

mathematics, in Babylonian mathematics. If Babylonian mathematics is genuine, we must recognise the position of mathematical acts performed on the objects of an authentic 'operative field', which is nothing other than the system of reciprocal relations through which these acts make sense of each other. These relations are always regulated, which is why such a field can be said to be normed in Desanti's sense, and not purely intuitive, since the rules make the act operative by objectifying it. Consequently, even at this very remote point in the development of mathematics, we observe what G. Granger refers to as the duality of the object and the operation. As M. Caveing[(9)] has demonstrated, there are at this early stage two types of properties: those of the *objects* of the field, which can prohibit certain acts such as subtraction or inversion, and those of the *operations* themselves, which primarily determine the chain of acts. Each object is said to manifest as an operative singularity, which, when authorising an operation with another or prohibiting it, is said thereby to possess such a property. This is the origin of the objectivity of the field: a property of the field is converted by the subjectivity of the mathematician-calculator into a property of the object. The properties of the field govern the operations that take place within it and therefore *also* how objects are produced. The process can only be carried out in accordance with the operative field's intrinsic laws, and the field inevitably manifests resistance and opacity.

2. The second thematic feature of depth would lie in the generality and abstraction of the objects in question. It is important to emphasise that these features merely express the conceptual content of mathematics. It is important not to confuse the objects' most apparent aspect — their abstract generality — with their status as products of conceptual thought.

From the outset, we should stress that the profound result enables us to reach a point of reference that is often assumed or expected as a general condition awaiting precise definition and formulation (which brings us back to conjectures). However, we would also argue that the profound result is an operative condition as opposed to a logical or set-theoretical foundation (or one of the more modern forms, such as category theory). There is a

[(9)] See M. Caveing, *La constitution du type mathématique de l'idéalité dans la pensée grecque.* Atelier de reproduction des thèses de l'université de Lille, 1982,1.1, p. 493-494.

foundation, if you like, but it is operative rather than logical or set-theoretical. Alternatively, we could say that this new condition lies latent in what Kronecker termed 'real mathematics' and must be extracted to provide a precise formulation.

Let's consider Kummer's invention of ideals, which remains within the realm of number theory. One result that can be established using Fermat's method of infinite descent (there are many other possible demonstrations) is the fundamental theorem of arithmetic (the 'FAT'): *any integer can be uniquely decomposed into a product of prime numbers.*[10]

This fundamental property of natural integers does not hold in other number systems. For example, in certain non-Euclidean rings of integers, such as the set of numbers in the form $a + b\sqrt{-5}$ usually denoted denoted $\mathbb{Z}[\sqrt{-5}]$, it is possible to obtain two essentially different decompositions of the same integer. Thus, $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. It was to reinstate this uniqueness of decomposition that Kummer developed ideal numbers, a paradigmatic example of a mathematical profound result. This decomposition led Kummer to create ideal numbers, which are a paradigmatic example of mathematical depth.

Due to specific historical reasons, Kummer initially worked with unusual integers known as cyclotomic. These integers are derived from numbers with "roots of unity", such as $a^n = 1$ (which can be represented geometrically as points for cutting out the circle). Kummer had to decompose these integers into their prime factors. In cases where these factors do not exist (as with 47), Kummer introduced so-called 'ideal' prime factors instead, while maintaining

[10] Or any integer can be uniquely put into the form of a product of a certain number of prime factors, possibly repeated (each of which may therefore be raised to a certain power). The so-called standard form of decomposition is obtained by choosing the ascending order and the uniqueness of the factorisation takes place up to the order, i.e., with a possible rearrangement of the factors. The FTA (Fundamental theorem of arithmetic) can be said to exhibit the structure of the natural numbers in relation to the operation of multiplication. It shows that the so-called prime numbers are the elements from which all the natural integers can be obtained by multiplication, carried out in every possible way. But when all these multiplications are carried out, the same number cannot appear in two different forms (one of the reasons, as we know, for not making 1 a prime number is that it would mean making an exception to the law of uniqueness of decomposition). Hence the fundamental place given to the theorem in modern expositions of number theory (see, for example, G. H. Hardy's classic treatise G. H. & E. M. Wright (*An Introduction to the Theory of Numbers.*)

most of their usual desirable properties. He set out these properties explicitly in his second memoir on the subject in 1847, including the laws of product, divisibility and so on. This resulted in a theory analogous to the theory of ordinary arithmetic. Except for the basis of the theory being the concept of prime divisor, the theory was analogous to ordinary arithmetic theory. Remarkably, however, Kummer did not immediately define this concept of divisor. Instead, he specified operatively the notions of divisibility by such a divisor (i.e., of one ideal number by another) and of congruence *modulo* such a divisor (since divisibility is defined by congruence conditions).

He operatively specifies the notions of divisibility by such a divisor (i.e., of one ideal number by another) and congruence *modulo* such a divisor, since divisibility is defined by congruence conditions. He defines congruence statements between a given cyclotomic integer and a prime divisor to explain what it means to say that the former is divisible by the latter. The result is a factorisation of the (cyclotomic) integer into prime divisors, rather than (cyclotomic) prime integers(11). A factorisation into cyclotomic integers only occurs if every prime divisor is a divisor of a cyclotomic integer. For example, the prime divisors of 47 are not divisors of cyclotomic integers. In this case, factorisation uniqueness fails. In general, there are two situations.

In cases where the number has an existing or current prime divisor, the two types of divisibility or congruence are identified: one is modulo the existing (or current) number, and the other is modulo an ideal prime divisor. The new definition coincides with the old one as soon as the latter is valid. However, the new definition is valid in cases where the old one was not. Congruence *modulo* the prime divisor of 47 is defined even if there is no cyclotomic prime integer that divides 47. Therefore, when there is no existing (or current) prime divisor, divisibility (and congruence) *modulo* a prime divisor becomes divisibility (and congruence) *modulo* an *ideal* prime divisor. Operative statements involving divisibility and congruence modulo a prime divisor retain their meaning even when there is no prime divisor.

Let us emphasize the following point: in this original procedure of 'idealization', the structuring of the condition remains constantly

(11)The relation in the ring of integers generated by the n^{th} roots of unity 'to be divisible by the n^{th} power of a k^{th} ideal prime' amounts to defining a *valuation* on the ring, but the notion would not be developed before the turn of the century.

operative.(12) The context of Kummer's invention is typically a context of use. Throughout these investigations, Kummer is guided by his experience with calculations on cyclotomic integers (or within the ring of such integers), and it is clear that, in general, his approach remains fundamentally analytical and inductive. This is particularly evident in his 1844 work, in which he attempts to factor binomial numbers (of the form $[x + a^j y]$, whose prime factors he knows to be the simplest within the arithmetic of cyclotomic integers.(13) Assuming, according to the classical method of analysis, that one such factor is known, he tries to deduce sufficient information from the consideration of particular numbers (induction) to make possible the explicit construction of prime divisors in a large number of cases. It is precisely the observation that there are cases where this construction method fails to yield the expected factor that leads Kummer to abandon the naïve assumption of unique factorization.(14) Inductive analysis thus essentially relies on the results of computations. The theory ultimately emerges as the product of this work, as an abstraction created from — or built upon — the manipulation of algorithms (especially those concerning divisibility).

With the process of theorization, the second feature mentioned earlier becomes apparent: in a profound result, there is the possibility of connecting scattered facts within a more general and abstract theory. This is clearly illustrated by Dedekind's theory of ideals — a set-theoretic, rigorous, and pure theory that played an essential role in the development of modern mathematics, and has often been commented on as such.

As with Kummer, the problem for Dedekind was to define 'ideal prime factors' in such a way that fundamental properties would remain valid. His first formulation, dating from 1871 (in the

(12) This is not quite a generalisation of the kind we have already made about numbers: an ideal number is not really a generalised species (or class) of complex numbers. Several different complex numbers determine the same ideal number, since cyclotomic integers that differ only by a multiple of unity have the same divisor.

(13) *De numeris complexis qui radicibus unitatis et numeris integris realibus constant*, 1844; Collected Papers, André Weil ed., Berlin, Heidelberg, New York, 1975, vol. I, p. 165-192. The reasons for choosing such numbers are both structural and historical: the first attempts to prove Fermat's theorem (notably by Lamé and Cauchy) involved factoring into this type of integer...

(14) An abandonment or a break made easier by the counter-examples known since Fermat (which can be found in Euler, Liouville or Jacobi) and the awareness since Gauss of the need to produce a demonstration in these matters.

Supplement to the second edition of Dirichlet's *Vorlesungen über Zahlentheorie*), is, from this point of view — as he would later acknowledge — nothing more than Kummer's theory presented in 'new clothing'.(15) Kummer had given a central role to the notion of the divisibility of one ideal by another, defining it via congruence conditions (two ideal numbers are equal if they are divisible by the same ideal prime numbers, with the same multiplicity). He described his "ideal prime divisors" by means of divisibility tests. The general case will have exactly the same definition.

Dedekind begins with the following point. The essential property of an ideal number (always understood here as a complex number) retained by Dedekind is whether or not it divides an actual number. One knows everything there is to know about an ideal number if one knows the (actual) numbers it divides. From this, he is naturally led to consider the set of integers of the field which are divisible by a given product of ideal prime factors, and to take this set as the representative of that product — this is the *ideal*. To each integer, one can thus associate the set I of its multiples. The problem of defining the ideal number is thereby replaced by the problem of defining the set I (called an ideal). Dedekind then characterizes it as a subset (he says 'system') of the collection of integers (in fact, a subset of a unital commutative ring A) closed under addition and multiplication by any element of the ring — these are the two classical properties, remarkably simple in form, which manifest here in the case of cyclotomic integers (more precisely, in the case of the integers of a number field K):(16)

1. The sum of two cyclotomic integers belonging to a given ideal also belongs to that ideal.

2. The product of a cyclotomic integer belonging to a given ideal with any cyclotomic integer belongs to the given ideal.

(15) *Cf.* §162 (end) of the X$^{\text{th}}$ Supplement to *Vorlesungen über Zahlentheorie* by P. G. Lejeune-Dirichlet, Vieweg Braunschweig, 1871; partially reproduced in *Gesammelte mathematische Werke* by Dedekind, R. Fricke, E. Noether & O. Ore eds, 3 vols, Vieweg, Braunschweig, 1930-1931-1932, vol. 3.

(16) The fact that the sets (of divisible integers) under consideration have these properties is fairly obvious. The converse is less obvious, and this is what Dedekind is most interested in proving: any subset of cyclotomic integers having these two properties is an ideal, i.e., the set of all cyclotomic integers divisible by A, for a certain divisor A. Dedekind thus got what he wanted: a characterisation of these ideal sets

We obtain a *bijective* correspondence — and even an *isomorphism* (with respect to the corresponding operations) — between ideal numbers and (sets of) ideals (apart from the case of the ideal $\{0\}$, which corresponds to no ideal number).[17] These two properties remain meaningful even if one does not already know that I consists of the multiples of a single number, and one can define an ideal as the subset of the ring possessing the relevant properties. Thus, we adopt a high level of conceptual generality, but — it should be noted — without preventing the establishment of a workable calculus. Dedekind succeeds in defining the rules of a calculus for entities that, while not elements of the ring, are entirely definable from it. These entities constitute subsets or parts of the ring, which is a decisive and essential novelty. One example is the definition of the *product* of two ideals. Consequently, all the fundamental propositions of the theory can be expressed in terms of ideals, such as the definition of a prime ideal without non-trivial factorisation.[18]

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(17) Dedekind's formulation fits entirely into (what we now call) the ring of integers of an algebraic number field, which can be any field K. It is very characteristic not only of Dedekind's practice, but also of his mathematical philosophy, to choose as a symbol, to represent all the elements of the field under consideration, the single letter K, without any indication of a coordinate, in preference to $K(a)$, which suggests a particular base. For Dedekind, it is the field that matters, and the explicit representation of the field in the form $\mathbb{Q}(a)$ for example (a being the solution of an algebraic equation over $\mathbb{Q}$), which is the result of an arbitrary choice, can, and therefore must, be avoided.

(18) An ideal is prime if it is neither the set reduced to zero nor the set of all cyclotomic integers of a given field and satisfies the following property: if the product of two elements belongs to the ideal, then at least one of the two elements must belong to it.

The success of Dedekind's theory is indeed explained by the level of generality and abstraction thus achieved. Its historical role in the development of abstract algebra is comparable to that of Leibnizian notation in the emergence of infinitesimal calculus. In support of this claim, it is sufficient to note that, to this day, scarcely any presentation of algebra fails to invoke this set-theoretic conceptual framework. Echoing the expressions most frequently used in subsequent discourse, one might say that the 'power' and 'elegance' of the theory have earned it unanimous admiration among mathematicians who have studied and built upon it. A single example will suffice: Hilbert adopted Dedekind's presentation in a celebrated memoir,(19) establishing the canonical form for all subsequent expositions of the theory.

4. As for depth — regarding which we are postponing our analysis of Dedekind's theory of ideals — the final component we believed we had identified was fruitfulness. The most compelling examples of profound results always reveal the opening of new fields of mathematical inquiry. This extension is itself made possible by the degree of generality attained and is inseparable from it. The theory of ideals would, once again, provide a remarkable example of this phenomenon.

The problem that arose after Kummer was indeed, as Dedekind's example shows, that of generalizing the theory. This generalization is based on a key notion: that of an integer within a field of algebraic numbers — the crucial point being the correct determination of the ring to which the theory of ideals applies. Now, there are several ways of determining this ring. The first, just mentioned, is Dedekind's approach in the second edition of Dirichlet's *Vorlesungen...* It consists of a direct generalization of Kummer's approach. The other two are: Dedekind's later version — which might be called his 'second theory" (in fact, there were several later versions)(20) — and Kronecker's

(19)D. Hilbert, *Die Theorie der algebraischen Zahlkörper*, Gesammelte Abhandlungen, vol. l, pp. 63-363.

(20)Disappointed by the lack of success with his first version, Dedekind wrote a long treatise which was published in French in the *Bulletin des sciences mathématiques*, under the title: *Sur la théorie des nombres entiers algébriques* (and almost identical in content to the version in the 3rd edition of Dirichlet's *Vorlesungen...* by Dirichlet, published in 1879). The final version of the theory, published in the 4th edition of the *Vorlesungen...* develops a completely new arithmetic of *modules,* subgroups of the additive group of complex numbers, with a very general theory of the negative powers of certain modules. The theory applies in particular to ideals, and makes it possible to give meaning to fractional ideals. An even different version would be based on his 'Prague theorem' (published in 1892, but found in 1887).

formulation.(21) Whether one adopts Dedekind's version (as was initially the case, notably by the most illustrious follower of this path, Hilbert) or Kronecker's (whose merits took longer to be fully acknowledged), the result is the same — and it is a remarkable one: the establishment of algebraic number theory, with its extensions, notably the development of class field theory. As for the fruitfulness of the theory initiated by Kummer, this indication may suffice, and we should not attempt to convey even a rough sense of it here.

5. We would now like to confront the results of the epistemological analysis outlined at the beginning — based on the suggestions from *Idéalités Mathématiques* — with a few simple historical facts. One of these is difficult to ignore: mathematics is made by mathematicians. There are profound mathematicians: they are the ones who discover profound results — they are the great mathematicians. Now, it is a fact that these individuals do not necessarily identify with foundational theorists, nor even with proponents of set-theoretic abstraction (or, today, category theory). At least two of the greatest mathematicians of the recent period can be cited here: Poincaré and Kronecker, both of whom either ignored or dismissed logic and set theory as unworthy of serious interest.(22) The set-theoretic formulation, introduced by Cantor and Dedekind, may prove to be misleading in this respect. While there is undoubtedly a mode of abstract set-theoretic thought, as exemplified by the

(21)*Grundzüge einer arithmetischen Theorie der algebraischen Grossen*, Reimer, Berlin, 1882; also in Journal fur Mathematik, Crelle, 92, 1882, pp. 1-122 and in Werke, K. Hensel, ed. 5 vols, Leipzig, 1895, 1897, 1899, 1929, 1930, vol. 2, p. 239-387. In his first publication, Kummer himself envisaged a generalisation of his theory to complex numbers of the form $x + y\sqrt{A}$, which would have linked it to the Gaussian theory of the composition of binary quadratic forms. He never returned to this question, generalising his theory in other ways.

(22)In a letter to Kummer, quoted by H. Meschkowski (*Problème des Unendlichen. Werk und Leben Georgs Cantors*, Braunschwick, 1967, p. 238): 'Like him [Kummer], I recognised the impossibility of relying on any kind of speculation and found refuge in the paradise of real mathematics...", and again: "In the field of mathematics, I find real scientific value only in concrete mathematical truths, or, to put it more succinctly, only in mathematical formulae. The history of mathematics has shown that only mathematical formulae last forever. The various theories on the foundations of mathematics (such as Lagrange's) have been set aside in the course of time, but Lagrange's resolvent is still there!' (ibid., pp. 238-239). Poincaré, in response to Russell, who felt that 'until a complete solution of our difficulties [i.e., paradoxes] is found, we cannot know with certainty what volume of mathematics will be left untouched...', replied: 'only Cantorism and logistics are called into question: real mathematics will continue to develop according to its own principles...". (*Mathematics and Logic*, 1906).

school of Cantor and Dedekind, it holds no special privilege and certainly no exclusivity, as some contemporary mathematical currents clearly demonstrate. In this regard, we would like to return to the case of Kronecker, which we consider to be exemplary.

Our analysis of Kummer's invention of ideals has enabled us to realise that depth was the result of an operational procedure rather than a theoretical determination of objectivity. This does not mean, of course, that the process of theorisation, at all levels of abstraction up to formalisation, is meaningless. Rather, it means that it does not appear to be transcendentally primary, in the sense that the condition of possibility would be provided by it. Using Granger's categories, we can say that, in mathematics, the operative precedes and grounds the object level (*l'objectal*). This is the essence of mathematical experience, producing content and being synthetic in Kant's sense — applying rules of operation to symbols and observing their results. In mathematics, there is something synthetic *a priori*; the *a priori* consists of determining the form of the results before they are actually observed.

From this point of view, Kronecker is a worthy successor to Kummer. Like Kummer, he focuses on how divisors are represented rather than on their intrinsic nature, explaining what it means to say that two representations correspond to the same divisor (or, in his terminology, that two divisors are 'absolutely equivalent'). The way he expresses himself is characteristic, as can be seen in particular in §15 of the *Grundzuge* and his definition of the divisor. On the one hand, he does not say that a divisor is an equivalence class of forms — two forms a_1 and a_2 being equivalent if and only if the congruence relations $\mod [a_1]$ and $\mod [a_2]$ coincide — as we would say today. On the other hand, he doesn't just provide definitions. He also explains how to perform calculations with divisors. Given a set of generators a_1, ..., a_k of algebraic numbers, he provides what we would call today an algorithm for determining whether a given element of the field belongs or not to the ideal generated by the numbers a_1, ..., a_k.

On the contrary, Dedekind does not have anything of the sort. For him, a definition of an ideal is only considered complete and satisfactory if it presents the ideal as an (infinite) set without any membership condition. Nevertheless, the abstraction of conceptual thought cannot be reduced to set-theoretic abstraction, let alone formal abstraction. Kronecker's approach to cyclotomic integers —

defined as expressions of polynomial form, with addition, subtraction and multiplication defined as they usually are on expressions of this (polynomial) type — is abstract and algebraic, and in this respect is very much in the spirit of contemporary algebra. In modern terms, the set of cyclotomic integers is the quotient of the ring of one-variable polynomials with integer coefficients, divided by the ideal generated by the polynomial $1 + a + a^2 + \cdots + a^{l-1}$. Its great virtue is that it emphasises the 'computational' rules of algebra for calculating the arithmetic of cyclotomic integers, pushing all other considerations into the background. This is a general feature of Kronecker's work. From the very first pages of *Grundzüge...*, he devotes a very significant section to the factorisation of polynomials with integer coefficients or with coefficients in a field of algebraic number. Although, as Edwards[(23)] notes, if his algorithms were probably not intended for practical application, they at least made it possible to state the problem clearly and provide an explicit starting point for the solution.

There's something else, too. Today, we realise to what extent the true purpose of the *Grundzüge* was far more ambitious than merely laying the foundations of algebraic number theory based on Kummer's theory of ideals. Kronecker's aim was different and far more significant: not merely the proper treatment of the fundamental problems of a theory of ideals — Dedekind's main subject — nor even an arithmetisation of analysis on the stricter basis of a doctrine of integers without actual infinity (*infini complet*), but something far greater. His aim was not an unfortunate attempt, driven by his erroneous philosophical thesis and his stubbornness in restricting the free development of analysis and mathematics within excessively narrow limits. In reality, Kronecker had a unifying vision: he aimed to develop a new branch of mathematics that would encompass both number theory and algebraic geometry as special domains. In other words, he sought to broaden the field of mathematics — rather than restrict it — by opening up a vast field for research: the development of an algebraic geometry based on arithmetic. This was a grandiose idea that gave true meaning to his project for a general arithmetic (*allgemeine Arithmetik*). This connection between number theory and algebraic geometry underpins what is aptly called Kronecker's 'programme'. According to some of the most renowned mathematicians of the past century, including E. Hecke, H. Weyl, C. L. Siegel and A. Weil, Kronecker's

(23) H. M. Edwards, *An Appreciation of Kronecker*, The Mathematical Intelligencer, vol. 9, no. 1, 1987, p. 35.

'algebraic' perspective, although it may appear limited by today's standards, actually encompasses all cases amenable to algebraic methods. As A. Weil argues, despite the insistence in modern algebraic geometry on the use of arbitrary base fields, there is a very real sense in which any theorem accessible to algebraic methods (as distinct from analytic or topological methods) can be considered a theorem on base fields that are either finite or fields of algebraic numbers. These are known as Weil's 'absolutely algebraic' fields. This is exactly what Kronecker's perspective means: 'Absolutely algebraic' fields are the natural base fields of algebraic geometry.(24)

Like Galois' example, Kronecker's shows that such a construction can be so broad and novel that it takes decades to understand it fully. Ultimately, this is an infallible sign that this conception deserves to be described as 'profound', a quality that all the great mathematicians who have studied it have agreed upon. When presenting Kronecker's significant results,(25) Hilbert lucidly highlights their importance in linking number theory with algebra and the theory of functions. Unfortunately, we have to quote this text without the commentary it would require:

> "It was Kronecker who gave us the theorem that any abelian field of numbers in the domain of rational numbers is generated by the composition of fields of roots of unity [...] After $\mathbb{Q}$, the simplest field is the field of quadratic numbers. The problem then becomes extending Kronecker's theorem to this field [...].
>
> Finally, I believe the most crucial step is extending Kronecker's theorem to cases where, rather than the domain of rational numbers or of imaginary quadratic numbers, the domain of rationality is any algebraic number field. I regard this problem as one of the most

(24) A. Weil, 'Number-theory and algebraic geometry', *Œuvres scientifiques*, 1.1, pp. 442-452. The same algebraic point of view makes it possible to propose, as A. Weil says, the study of algebraic geometry on a ring, for example the ring of integers, or that of the integers of an algebraic number field, or that of the integers in a p-adic field (local ring of p-adic integers): a programme that Grothendieck's theory of schemes has largely fulfilled. It also made it possible to develop the theory of abelian functions, also studied by A. Weil, as part of algebra.

(25) The central result is the theorem-conjecture known as the *Jugendtraum* (Kronecker's childhood dream, as he called it in a letter to Dedekind), on abelian extensions of quadratic imaginary fields (reduced to sub-fields of cyclotomic fields).

> profound and important in the whole theory of numbers and functions".[ii]

Thus, the Kronecker episode offers us a prime example of the 'metastable depth', as *Les Idéalités mathématiques* put it, 'in which the regulated yet non-mechanical process of mathematical creation appears to unfold'. As well as this epistemological lesson, there is also a historical one. Firstly, Dedekind's mathematical style and approach to foundations made Kronecker's work more difficult to read than it actually was. Secondly, seeking to make the works of Kummer and Kronecker accessible to students by reformulating them in his *Zahlbericht*, Hilbert was essentially trying to render them obsolete. Thirdly, no one is unaware, especially in France, of the extent to which the Bourbaki school was influenced by the abstract algebra style of Dedekind and Hilbert, both in its presentation of mathematics and in its approach to foundations in general. The example of the ring and the ideal of a ring, which are conceived first and foremost as the underlying sets (Dedekind's 'systems') which are generally infinite, can be taken as an example. Lastly, we are aware of the extent to which these conceptions also influenced the French mathematical philosophy of Cavaillès. Cantor and Dedekind were so successful that the term 'foundations of mathematics' was used long after their time, and until recently, to refer to set theory, particularly infinite set theory. In today's age of calculating machines, the ability to 'test' hypotheses and calculate data with unprecedented speed and ease has not only changed the way we solve problems, but also the way we think about them. It is important to draw historical conclusions from this. The success of Dedekind and Hilbert also masked a failure. There was much more in Kronecker's work than even the greatest of future generations of mathematicians could grasp. Only his memoirs, which offer a glimpse of a possible revival, can convey their profound nature.

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[ii] Alain Michel quotes a passage from Hilbert's twelfth problem in the following French translation: D. Hilbert: *Sur les problèmes futurs des mathématiques. Les 23 Problèmes*. "Extension du théorème de Kronecker sur les corps abélien de rationalité algébrique quelconque". Paris Ed Gabay, 1990. Problème XII p.31, 32. For the German edition see *Gesammelte Abhandlungen dritter band*, Chelsea Publishing Company, New York (1965) p. 311, 312. [Ed.]