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The philosophical significance of algebraic geometry(*)

Alberto Peruzzi

Abstract. This paper explores the philosophical significance of algebraic geometry by addressing Federigo Enriques' question on the relation between logic and intuition. Through a historical and conceptual analysis, it traces the transition from the Italian school of algebraic geometry to the abstract frameworks developed by Grothendieck and Lawvere. The article highlights how key categorical notions — such as schemes, sheaves, and toposes — transform the interplay between geometry and logic, allowing logical principles to be internalized within geometric structures. It argues that the philosophy of mathematics cannot be reduced to meta-mathematical reflection alone, since algebraic geometry itself generates conceptual innovations with direct philosophical import. Ultimately, the paper shows that algebraic geometry reshapes the foundations of mathematics by dissolving the separation between formal rigor and spatial intuition and providing support to the reasonable effectiveness of "conceptual mathematics". This approach not only provides an answer to Enriques' question but also defines a new sense for the foundations of mathematics, where logical principles are intrinsically linked to the geometric structure of a mathematical universe.

Keywords. Algebraic Geometry, Logic, Topos, Adjoints, Foundations of Mathematics, Enriques, Grothendieck, Lawvere, Conceptual Mathematics, Philosophy.

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§ 1. — An overview.

> [...] the usual question, whether mathematics should rather educate *intuition* or *logic*, is vitiated by an imperfect view of the value of teaching. In fact, the assumption of this question is that logic and intuition can be separated as distinct faculties of intelligence, whereas they are rather two inseparable aspects of the same active process, which refer to each other.

In these words of pedagogical intent, taken from an article by Federigo Enriques, published in 1921, [11], the necessity of overcoming the Kantian dichotomy within the plane of the a priori, between mathematical intuition and logic, is envisaged. Enriques did not subscribe to the reduction of mathematical truths to logical truths, but neither did he want to re-found logic through the adoption of constructive constraints based on a special type of intuition, so his idea diverged as much from the logicism espoused by the neo-positivists(1) in the 1920s as from intuitionism.

Enriques' invitation to recognise the *essentially composite* nature of mathematical knowledge raises an immediate question:

(*) *Can the link between logic and intuition be specified mathematically*?

In order to reduce the vagueness of the question, it is necessary 1) to make explicit what is meant by 'logic', 2) to make explicit what is meant by 'intuition' and finally 3) to specify the supposed link: condition 1) requires an appropriate formal language in which to express the logical principles; condition 2) requires, in order to keep to what Enriques intended, that intuition be anchored to 'spatiality' and, possibly, to its characters that are expressible in terms of algebraic geometry; condition 3) requires a mathematical formulation of the link between logical structure and geometric structure. Enriques was especially aware of the algebraic approach to logic, and he was obviously aware of the multiplicity of geometries, but equally obviously could not envisage a correspondence between logical properties and topological properties.

(1) In the current English lexicon of the philosophy of science, the tern 'logical empiricists' is much more frequent than the terms 'logical positivists', 'neo-positivists' or 'neo-empiricists', although strictly speaking there is no empiricism that can be called 'logical', thus each of these three less frequent terms seems to be preferable.

Yet, a *positive* answer to the question (*) came precisely from the subsequent development of the area of mathematics in which Enriques obtained relevant results, namely, algebraic geometry, in which algebraic varieties, defined as sets of points which are (common) zeros of (systems of) polynomial equations, are studied. This response took shape mainly thanks to the pioneering ideas of Alexander Grothendieck and Bill Lawvere, as well as, of course, the contribution of many mathematicians who made use of those ideas. But the theoretical path to arrive at such an answer was a long one, and in order to understand its meaning, it is appropriate to retrace its main steps.

Together with Corrado Segre, Guido Castelnuovo and Francesco Severi, Federigo Enriques was one of the major representatives of the "Italian school of algebraic geometry", [6] whose initiator is recognised in Luigi Cremona, and for a long period of time, from 1885 to the year of his death in 1946, Enriques gave great impetus to research in this area by contributing to the classification of algebraic surfaces in terms of invariants with respect to birational transformations — a result that ensured Enriques international notoriety. His prestige then seemed to be undermined by the need for greater rigour (which Segre had already called upon the 'volcanic' Enriques), by the use of topological methods (Solomon Lefschetz) and abstract algebraic methods (Emmy Noether), which were put to use after 1945 by the 'French school' (Jean-Pierre Serre, André Weil, Alexander Grothendieck), to the detriment — at first sight — of intuition. But the contribution of the Italian school was later re-evaluated by further advances in algebraic geometry, thanks especially to David Mumford and Igor Shafarevich, [30], [43]. Some of the concepts employed in 'abstract' algebraic geometry from the 1940s onwards are due to the work of Oscar Zariski, who had studied in Rome in the 1920s under Castelnuovo before emigrating to the United States. It is no coincidence that Zariski has been described as 'the last of the great classical Italian geometers and the first of the great modern geometers'.(2) It was, in fact, Zariski who in his 1935 treatise, [44] set himself the task of making rigorous proofs of the results obtained by the Italian school.

If the numerous works that Enriques dedicated to the classification of algebraic surfaces have justified his fame in the mathematical sphere, outside of that sphere his notoriety was due above all to

(2) [1] p. 171. For a biography of Zariski, see [31].

the breadth of the reflection he conducted on science, [9] seen from a historical perspective, as Ernst Mach and Pierre Duhem already suggested. More specifically, in such a perspective are (A) his interest in the history of mathematics and, more specifically, of logic, [8], (B) his commitment to innovating the teaching of mathematics, as witnessed in [5], (C) his proposal of a framework within which a fruitful dialogue between science and philosophy could take place, once distance is taken from (anti-scientific) idealism and (anti-philosophical) positivism, [10].(3)

Leaving aside the manner in which Enriques articulated his pedagogical commitment to (B), some minimal indications about (A) and (C) are appropriate.

Ad (A). While Mach and Duhem had attributed theoretical value to the analysis of the historical roots of mechanics, Enriques was inclined towards a reconstruction that was not 'thesis-based', in order to document a more articulate relationship between philosophy and science, and specifically between philosophy and mathematics, throughout history. His collaboration with a young physicist, Giorgio de Santillana, who was passionate about the origins of science in Greek thought, should be seen in this light. Enriques ensured that Santillana was called to teach in Rome and with him he began to write a *History of Scientific Thought*, which stopped at the first volume (1932), [12] because Santillana, from a Jewish family like Enriques, emigrated to America two years later, where he published, in English, those works of a historical-scientific nature that made him famous.

Just as Zariski deserves the merit of having contributed, among the first, to ferrying the research of the Italian school of algebraic geometry towards 'modernity' (in the sense of Bartel van der Waerden), Santillana can be credited with having given substance to Enriques' ideas in the field of the history of science. It is a pity that the various texts on the history of science that appeared in English after the Second World War and inspired by the neo-positivist conception, when they refer to Santillana's investigations, forget his mentor, even though the jointly signed *Compendium of the History of Scientific Thought was* published in 1937.(4)

(3) To foster this dialogue, Enriques organised the Fourth International Congress of Philosophy in Bologna in 1911, in the hostility of a significant part of the Italian philosophical milieu.

(4) Little appreciation of Enriques' impetus for investigations in this field can also be found in original Italian-language texts bearing a similar title.

Ad (C). The flowering of studies on Enriques' thought over the last forty years, thanks in particular to Ornella Pompeo Faracovi, [40] has made it possible to highlight multiple aspects of the dialogue between science and philosophy as hoped for by the mathematician from Livorno and, after a long period in which his epistemological perspective had been dismissed as an unlikely combination of psychologism and physicalism, it is now possible to re-evaluate those ideas in the light of the centrality that cognitive sciences have acquired. That dialogue stemmed, for Enriques, from the need to analyse concepts taking into account their genesis and the multiplicity of their potential developments, in view of a 'scientific philosophy' finally capable of being translated into a teaching methodology enriched by historical-epistemological awareness. Enriques sought support in vain in the Italian philosophical milieu. His project soon clashed with the dominant idealism, as well as with a crude positivism widespread among scientists. It is significant that already in the 1930s Enriques also distanced himself from the neo-positivism configured by the Vienna Circle. If the importance of the logical analysis of language escaped him, he avoided the risk of a total 'linguisticisation' of philosophy that, in the footsteps of Wittgenstein, Moriz Schlick and Rudolf Carnap had outlined.

Unfortunately, Enriques failed to elaborate a clear framework of ideas as an alternative to old and new positivism. The project of a 'scientific philosophy' was not specified by him even limited to the mathematical sphere; and in any case, any such attempt would have hinged on the interconnections between geometry and algebra rather than on those between logic and set theory,(5) while to the mathematicians who in the early 20th century paid attention to 'foundational' problems, such an attempt would have appeared retrograde: the idea of founding mathematics on a basis of incontrovertible certainty was animated by the conviction that this basis, provided by a logical and set-theoretic approach, removed any appeal to geometric intuition.(6)

If the foundational framework has changed, it is mainly due to the development of concepts introduced by Alexander Grothendieck, when he used category theory to solve problems in algebraic geometry. It was, however, Bill Lawvere who first

(5) A concise overview of Enriques' philosophical ideas is provided in [37].

(6) Yet Gottlob Frege himself, in manuscripts of 1924-25, returned to assign geometric intuition a status irreducible to logic.

realised that those concepts were relevant to the foundational question and prefigured a direct link between logic and geometry, such that the demands of constructiveness picked up by 'intuitionistic' logic were *embedded* in the structure of an arbitrary topos.

The purpose of the following pages is to show that the development of abstraction in algebraic geometry, first with algebraic methods (ring theory) and then with categorical methods, allowed Lawvere to state that "Algebraic Geometry = "Geometric Logic", [21] and that this specific unification of logic and geometry makes a first, *partial*, positive answer to the question (*), in view of a more comprehensive answer, once the task of formulating in mathematical terms the characters of the *kinesthetic patterns* that underlie the understanding of the meaning of any proposition has been completed, [33].

After a reference, in §2, to the idea of a "conceptual mathematics" (from Emmy Noether to Grothendieck and finally to Lawvere), the salient steps along the path from algebraic geometry to logic will be described, highlighting five "key concepts" that emerged along this path, which are endowed with foundational scope and, together, identify part of the philosophical meaning that can be attributed to algebraic geometry, without assuming that they exhaust it: more specifically, § 3 will give a concise idea of the path that led Grothendieck to the notion of topos, and § 4 will describe both the salient points of the connection with logic discovered by Lawvere and the new meaning that Lawvere gave to the foundations of mathematics. In order to appreciate, by contrast, the distinctive features of this approach, the main ways in which the relationship between geometry and philosophy has been historically configured will be recalled in § 5. Finally, in § 6, some objections to this approach and some further lines of development will be mentioned.

§ 2. — The idea of conceptual mathematics.

Between Enriques' approximate philosophical framework and the specificity of his work in algebraic geometry, a profound gap remained. The other mathematicians of the Italian school of algebraic geometry did not share his need for dialogue with philosophy, much less were they interested in filling the gap. But, like Enriques, they avoided engaging in the debate on foundations that had originated from Cantorian set theory, even though in the preface to the first volume (1893) of the *Grundgesetze* Frege had already stigmatised the widespread opinion that any philosophical consideration is out of place in mathematics, and vice versa. Conversely, the task of consolidating the *specific foundations* of an extended algebraic geometry was little felt by those interested in foundations in general.

The line of logicism inaugurated by Frege had been corrected by Bertrand Russell with the (ramified) theory of types in order to avoid paradoxes. Opposed to this line, far removed from mathematical practice, were the line promoted by Ernst Zermelo, oriented towards the formulation of a system of axioms that expressed the principles for generating the universe of sets, and the metamathematical line of David Hilbert's formalism. There was no lack of critical positions towards the 'Cantor's paradise' — to use Hilbert's expression — such as those expressed by Henri Poincaré (partly taken up by Russell) and Luitzen E. Brouwer, respectively, but these positions prevented the conservation of all the results obtained in 'classical' mathematics. Certainly, Enriques was unwilling to pay such a price, but he did not adhere to logicism, nor did he commit himself to follow the lines of Zermelo or Hilbert, perhaps in the idea that (algebraic) geometry was safe from those controversies.

The Hilbertian call for rigour also had repercussions in this area, however, thanks to a specific innovation: the notion of ideal in the theory of rings. It was Emmy Noether who gave impetus to rigour through the development of 'abstract', axiomatically exposed methods. But among the many possible ways of *mathematical abstraction*, how to make the selection? Noether's idea was that the validity, as well as the fruitfulness, of an abstraction depends on its exportability to areas of mathematics other than that in which it was originally motivated. This idea, summarised, in German, by the Russian topologist Pavel Alexandroff, under the term *begriffliche Mathematik* (conceptual mathematics) was picked up by Grothendieck and later taken up by Bill Lawvere and Stephen Schanuel, who not surprisingly entitled their introductory text on category theory, [22] *Conceptual Mathematics*.

In algebraic geometry, the rigour provided by an axiomatic approach became all the more pressing the more one abstracted from the reference to curves and surfaces immersed in ordinary three-dimensional space. In the projective environment, one could still appeal to geometric intuition, but once out of this environment, one had to privilege algebra over geometry. In fact, algebraic methods fuelled a growing abstraction, freeing algebraic geometry from the privilege accorded to $\mathbb{C}$, the field of complex numbers, and envisaging its development over any algebraically closed field K.

As already noted, the specific task of providing a solid foundation for algebraic geometry was little perceived by those concerned with the foundation of mathematics as a whole: arithmetic came before algebraic geometry, so the focus was on the notion of (natural) number and from this shifted to the notion of set and its axiomatic characterisation, focusing on the axiom of choice, the principle of comprehension, and proofs by *reductio ad absurdum*, and not on principles and methods peculiar to algebraic geometry.

The flowering of the French school of algebraic geometry from the 1940s onwards brought with it a further increase in abstraction, which reached its peak with Grothendieck. The notions he introduced were aimed at solving various specific problems but acquired a broader sense thanks to Grothendieck's efforts to give algebraic geometry a new, systematic structure,(7) to the point of acquiring, with Lawvere, direct relevance to the foundations of mathematics in its entirety, in a way that suggests how to fill the gap left by Enriques.

Before mentioning the features of Grothendieck's approach that are relevant for this purpose, it is possible to get an idea of his working method, in accordance with his use of category theory, through a metaphor he uses in *Récoltes et Semailles*, [17] when he likens a mathematical problem to a walnut to be cracked: there is the method of putting the walnut on the anvil and beating on the shell with a hammer (the use of a common nutcracker is a more convenient variant), but there is also another method, which Grothendieck made his own and which he describes as follows: "on plonge la noix dans un liquide emollient, de l'eau simplement pourquoi pas, de temps en temps on frotte pour qu'elle pénètre mieux, pour le reste on laisse faire le temps".(8)

(7) As reflected in the structure of EGA, [15], and the pioneering research conducted in the seminar held by Grothendieck at the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette (Paris), with particular reference to SGA 4, [16]. The legacy of those ideas was then framed in reference texts such as [43] and [18].

(8) Again to illustrate how an increase in abstraction allows 'concrete' problems to be solved, Grothendieck made use of a similar metaphor: that of the high tide, whereby water penetrates a substance that seemed resistant to penetration.

Grothendieck adds, that in the development of new concepts for algebraic geometry he was guided not by the intention of obtaining a particular domain-specific result but by the idea of developing concepts and methods that in the future could also be reused in a different and broader domain, beyond that in which they had been used to open the nut, i.e., to provide a solution to a problem — perfectly in line with the Noetherian idea of *begriffliche Mathematik.*

Putting the nut in the water meant using categorical concepts to solve algebraic geometry problems by immersing them in an 'abstract' universe-of-discourse, as is, for example, a topos. This same method proved fruitful even beyond the original scope of the problems Grothendieck wanted to solve: it was Lawvere who grasped the fruitfulness of the concepts and methods introduced by Grothendieck in the foundational sphere, going so far as to identify the properties that define any topos, understood by Grothendieck as a 'generalised space', of which the universe of sets is only a particular case.

§ 3. — **Towards the concept of topos.**

The framework in which the Italian school of algebraic geometry moves is identifiable with the complex projective space of dimension *n.* It was Luigi Cremona who had adopted an algebraic approach to projective geometry by operating a generalisation of projective transformations that consisted in considering the class of all birational transformations (invertible rationals, expressed in coordinates) of a space, in order to achieve a classification of specific geometric entities in terms of what is *invariant* with respect to these transformations. Guido Castelnuovo and Federigo Enriques focused their attention on families of curves on a surface and, in particular, Enriques was deeply involved in the project of their classification. Within a few decades, a vast body of knowledge, notions and methods was thus created that needed a stable, rigorous structure. This need was accompanied by another: that of achieving results of a more general scope. This meant no longer confining oneself to the field of complex numbers.

Oscar Zariski and André Weil's pioneering investigations in this direction established close links with topology and number theory. Zariski defined a topology (called 'Zariski's topology') on the spectrum Spec(R) of a commutative ring R, i.e. on the set of prime ideals of *R:* the closed sets $Z(I)$ are the sets of prime ideals containing an ideal *I*. Given a ring of polynomials in a fixed number of indeterminates with coefficients in *R*, the ideal generated by an irreducible polynomial is prime, and the importance of the spectrum properties is immediate because an algebraic variety consists precisely of the zeros of ideals in a ring of polynomials.(9) Weil advanced a series of conjectures in number theory in which the analogue, for finite fields, of the Riemann Hypothesis found expression. The proof of Weil's conjectures was the challenge that Grothendieck took up through the development of sheaf cohomology and through the introduction of the concept of a *scheme* as a generalisation of the concept of an algebraic variety.

A *scheme* is, in fact, given together with its covering by affine schemes. In a general topological sense, it is a space such that each open *U* is associated with a commutative ring and the spectra of these rings can be 'glued' together. With reference to the affine case, Ciro Ciliberto summarises the idea as follows: "an affine scheme *X* is, in short, assigned in the affine space A^n by giving its equations i.e., by giving the ideal I_X of the ring *S* of all polynomials in *n* variables (i.e., the regular functions on A^n) that have their zeros on the scheme" and "just as a variety is covered by openings that are affine varieties that conveniently glue together, a scheme is covered by affine schemes that glue together", [5]. Exploiting the properties of the bundle of regular functions on a scheme *X*, defined as a *structural bundle* of *X*, Grothendieck went on to consider no longer a single sheaf but the entire *category of sheaves* on a space.(10)

The notion of a sheaf had been introduced by Jean Leray in some papers from 1945-47 and then clarified in 1947 by Henri Cartan, to whom we also owe the definition (in 1950) of *espace étalé* formed by a given space *X* together with the collection of spaces that are locally homeomorphic to *X*.(11) It is precisely by generalising this notion that Grothendieck successfully tackles the challenge posed by Weil's conjecture. The generalisation is achieved by abstracting certain characteristic properties of a generic topological space *X*, without referring to the elements, "points", of *X*, but considering only the algebraic structure of the "parts" of *X*, i.e., to the

(9) For a clear and comprehensive overview, see [7].

(10) A clear reconstruction of the path followed by Grothendieck to prove Weil's conjectures is offered in [29].

(11) For those unfamiliar with the notion of a bundle, the concrete examples described in [42] may help.

lattice $O(X)$ of its opens (in effect, a complete Heyting algebra). The underlying idea was that what matters of such a structure can be expressed in terms of the coverings of each open U: instead of limiting oneself to the partial order associated with the inclusion (between opens), one could have recourse to the properties of appropriate covering families of morphisms having U as codomain and study their transfer to any open of U.

Grothendieck realised that the notion of covering used to define a sheaf on a topological space could be expressed in purely categorical terms and thus arrived at the notion of a *site* as a category endowed with what later came to be known as 'Grothendieck topology' and the notion of *topos* as a category of sheaves on a site. Instead of $O(X)$ as a *poset category*, one takes a more general category and asks, firstly, how to transfer a structure defined on a particular object, to another object, and secondly, in the case of these objects being categories, one asks how to transfer, by means of a functor, the given structure from one category to another. This is how we arrive at a generalised notion of *base change.*

To define this notion, category theory was indispensable. Indeed, in order to make a base change within a category **C**, the category must first have pullback. Given a morphism $f : A \to B$ in **C**, f induces a functor $f^* : \mathbf{C}/B \to \mathbf{C}/A$ that associates to any $g : X \to B$ the morphism $f^*(g) = g' : A \times_B X \to A$ obtained via pullback of g along f. The functor f^* is contravariant (as is the cohomology functor from topological spaces to abelian groups, while the homotopy functor is covariant). The object $X' = A \times_B X$ is said to be obtained 'by change of base' from B to A. In particular, fixed B as a base and taking an element $b \in B$, the *fibre* of g on b is $x \times_B \{b\}$ via the inclusion $\{b\} \subset B$.

Considering a category of sheaves of sets on **C**, a sheaf F is similarly transported. The case of interest for algebraic geometry is when B is a scheme and F is a sheaf of abelian groups. A 'base change theorem' then states what remains unchanged (up to isomorphism) by base change; the validity of the theorem obviously depends on the properties of **C**.

A *Grothendieck topos*, as a category of sheaves on a ('small') site(12) has finite limits, so it is a category **E** with a terminal object,

(12) A proper class is a collection sufficiently "large" to be in one-to-one correspondence with **Ord**, the class of all ordinals. A collection is "small" if it is not a proper class. If the use of the term 'set' is made in opposition to 'proper class', it suffices to say that a class A is proper if there is no class B such that $A \in B$; if there is,

equalizers and pullbacks, thus it has finite products too — the Cartesian product $A \times B$ of two objects A and B being obtained via pullbacks over the terminal. Furthermore, **E** is Cartesian closed, so for each A and B in **C**, it has exponentials B^A,[(13)] which behave in the expected way, i.e., the exponential objects 'internalise' the functor *Hom* from $\mathbf{C}_{\mathrm{op}}\times$ **C** to **Set**, which maps each pair $\langle A, B\rangle$ of objects of **C** (locally small) to the set of **C**-morphisms from A to B and maps functions to morphisms in the obvious way, so that $Hom(A \times B, C) \cong Hom(C, B^A)$. In the usual lexicon, B^A is 'the space of functions' from A to B. The existence of such an object is trivial in **Set**, whereas for other categories (e.g. when considering a category of spaces) it is not. Finally, in a Grothendieck topos **E** the duals of the limits listed above are also present, so **E** has initial object, coequalizers and pushouts (and hence co-products).

Let us now consider the *Sub* functor: $\mathbf{C}^{\mathrm{op}} \to$ **Set**, which assigns to each object B of **C** the set of its subobjects, where each subobject is the equivalence class (up to isomorphism) of all $\langle A, B\rangle$ such that m is a monomorphism $A \rightarrowtail B$. In general, it is not certain that an object that internalises the *Sub* functor exists in **C**. It was Lawvere who realised that in each Grothendieck topos **E** there is such an object, i.e., **E** has a *subobject classifier* Ω: for each $A \rightarrowtail B$, there is one and only one morphism $\chi_m : B \to \Omega$: such that m is the pullback of a special morphism t (*'true'*)*:* $1 \to \Omega$. At this point we define an *elementary* topos as any Cartesian closed category with finite limits which has a subjective classifier.

Every Grothendieck topos is an elementary topos, but not vice versa. A necessary and sufficient condition for a category with finite limits to be (equivalent to) a Grothendieck topos is established by Giraud's Theorem, which, when referring to a category **E** that is an elementary topos, reduces to the following two requirements:

- **E** has all colimits indexed by sets (i.e. small colimits, and in fact small arbitrary co-products suffice);

A is a set. In this way, small sets are simply sets, i.e. classes that are not proper. For many aspects of category theory, it is sufficient to work with *locally* small categories, i.e., such that the collection $Hom_{\mathbf{C}}(A,B)$ of morphisms between any two objects A and B of **C** is a set. In Grothendieck's sense, small sets are those that are elements of a Grothendieck universe U (a collection of sets transitive downwards and closed with respect to pairs, power sets and arbitrary unions indexed in U).

[(13)] In general, given two objects A and B in a category **C**, the exponential, when it exists, is the object B^A which corresponds to the 'function space' from A to B.

- **E** has a (small) set of generators.[14]

Lawvere and Tierney pointed out that a Grothendieck topology is *logically* expressible as a modal operator on Ω, i.e., as an endomorphism $j : \Omega \to \Omega$, which expresses the idea of *local truth* and is characterised by the following properties: $j \cdot true = true$, $j \cdot j = j$ and $j \cdot \wedge = \wedge \cdot j \times j$.

In a category of sheaves, a morphism from the terminal, taken as the base on which a sheaf may be defined, to any object F is a *global section* of F. The global sections of Ω (the truth values) are in one-to-one correspondence with the subobjects of the terminal, via Cartesian closure: in fact, since $Sub(B) \cong Hom(B, \Omega)$, we have $Sub(1) \cong Hom(1, \Omega) \cong \Omega^1 \cong \Omega$. In **Set**, Ω reduces to the Boolean algebra $2 = \{0, 1\}$ where 0 stands for *False* and 1 for *True*, so there are only two morphisms from the terminal to Ω. But this does not apply to a topos of non-constant objects, i.e. having a base other than a singleton.

The subobjects of 1 in an arbitrary topos turn out to form not a Boolean algebra but a Heyting algebra, which from Ω is transferred to $Sub(B)$, for each object B in the topos. The corresponding logic is intuitionistic (higher-order), now expressed in semi-equational form (because the composition of morphisms is not everywhere defined). In a topos of sheaves on a base space X, the terminal is the identity sheaf on the base space, so $Sub(1)$ can simply be thought of as $O(X)$. Classical logic is re-obtained as a special case: for example, if X is discrete, $O(X)$ is a Boolean algebra and the intrinsic logic of the topos then becomes classical logic, thus with the principle of excluded middle.

Within this framework, it is possible to refine the notion of *truth*. Tarski had specified this notion in set-theoretic terms, thus within **Set**. Likewise, for a language that can be interpreted in any other elementary topos, the notion of truth admits a formal semantics, but now the semantics is functorial, thus it is *more constrained* and at the same time *more general* than that expressed in set-theoretic terms. André Joyal's contribution to the clarification of sheaf semantics and, more generally (when the gluing condition is removed) presheaf semantics is referred to as 'Kripke-Joyal semantics'.

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(14) The conditions provided by Giraud's Theorem are independent of the specific properties of the base site, if any, and indeed there may be sites **C** and **C′** that are not equivalent (as categories) but are such that $\mathbf{Sh(C)}$ is equivalent to $\mathbf{Sh(C')}$.

The clarification of such semantics has been a fundamental achievement for both mathematics and philosophy: as it was acknowledged that Tarskian semantics and relational, or 'possible worlds', semantics (due to Kripke et al.), are indispensable tools for analysing the link between *truth* and *meaning*, a similar acknowledgment is required for categorical semantics, developed to interpret theories (first-order or higher) and investigate the class of models no longer limited to those in **Set.**

§ 4. — From algebraic geometry to logic.

While the importance of Grothendieck's achievements in algebraic geometry was immediately recognised, the importance of the change in language and method that loomed for logic and set theory with the use of notions from category theory took longer to be recognised, because of the difficulty to identify the foundational scope of the concepts developed in algebraic geometry, and the very idea that they could have philosophical relevance was even more difficult to accept.

The path from algebraic geometry to a core of categorical notions and methods as the nucleus of a new 'grammar' of mathematical language in its entirety has not been the easiest. The idea that certain categorical constructions put to use in a particular sphere such as algebraic geometry had significance for long-debated issues in the philosophy of mathematics met with sceptical, if not hostile, reactions, and attempts were made to absorb the results through appropriate 'translations' that safeguard the set-theoretic paradigm.

The abstraction required by the immersion of the 'nut' in an environment (category) of variable objects on a given base, and then by the variability of this same base, was not merely aimed at greater generality but oriented towards *conceptual mathematics.* It was this abstraction *that* led Grothendieck to the results that earned him the Fields medal in 1966 and allowed one of his students, Pierre Deligne, to prove Weil's Conjectures.

Grothendieck's use of categorical language went well beyond the horizon that had motivated the formulation, in 1945, of the concept of category by Saunders Mac Lane and Samuel Eilenberg, which was not due to a generic trend towards abstraction, but rather to the need to give a uniform structure to the concept of 'natural transformation', relative to the correspondence, which today we call 'functorial', between topological spaces and homotopy and homology groups. With hindsight, the definition of a category seems a truism, and not even Mac Lane and Eilenberg expected that their toolbox would serve to *solve* problems of algebraic geometry, let alone formulate a theory capable of standing as a candidate for the foundation of mathematics.

There were those who understood category theory as a *lingua franca* and those who spoke of it (ironically) as an *abstract nonsense*. But a language is not enough to prove theorems by means of which one then solves problems that are anything but trivial, and an *abstract nonsense* does not produce the specific use that Grothendieck made, in algebraic geometry, of categories of sheaves, and more specifically sheaves of rings and sheaves of groups on a space X, and then on a site C. Such a category is referred to as a 'Grothendieck topos'.(15)

It was Lawvere who realised the foundational potential of the conceptual framework developed by Grothendieck, after having shown in his 1963 doctoral thesis how, in purely categorical terms, the concept of *algebraic theory* could be made independent of linguistic presentation, thus providing new foundations for universal algebra. The following year, Lawvere axiomatised set theory without making use of $\in$. It only remained to deal categorically with logic in order to shape the new approach to foundations. For the propositional calculus, as well as for the lambda-calculus, a cartesian closed category would suffice, as Lawvere showed. But how to express quantification algebraically? There were already cylindric algebras and monadic and polyadic algebras, but apart from moving in the Boolean domain (only suitable for classical logic), the techniques developed were far from providing a manageable tool capable of expressing the comprehension principle as well.

As already mentioned, Lawvere realised that in every Grothendieck topos there is a special object: a *subobject classifier*, Ω, so that Grothendieck himself referred to it as a 'Lawvere object'.(16) Finally, in 1970, Lawvere and Miles Tierney succeeded in extracting a set of properties of a Grothendieck topos and used

(15)In what follows, we will use 'topoi' as the plural of 'topos' in Greek, rather than 'toposes', though Grothendieck intended the term 'topos' just as an acronym for 'TOPOlogical Space', to refer to a generalised space.

(16)The definition of this concept is provided in § 4.

these properties to introduce the notion of *elementary* topos (the definition of which later turned out to be reducible to that of a category with finite limits that is Cartesian closed and possesses a subobject classifier).

Compared to Mac Lane's original framework, the turning point, essential for both Grothendieck's and Lawvere's work, was the concept of *adjunction* between functors,(17) introduced by Daniel Kan. (For what concerns the origin of Kan's discovery, Jean-Pierre Marquis informed me that Kan, after attending a seminar by Eilenberg, realised that the special symmetry manifested through the relationship defined by Eilenberg between the tensor product and the hom-functor for two proper categories could lead to a more general notion). Grothendieck immediately grasped the importance of the concept and arrived at the formalism of the "six operations" with reference to a morphism $f : X \to Y$ of schemes. Such a morphism induces precisely three pairs of adjuncts between the categories of (abelian) sheaves $\mathbf{Sh}(X)$ and $\mathbf{Sh}(Y)$: $\langle f^*, f_*, f_!, f^!, \otimes, Hom\rangle$, where f^* is the inverse image functor, f_* the direct image, $f_!$ and $f^!$ respectively the direct and the inverse image with proper support, and $\otimes$ is the tensor product as left adjunct of the inner *Hom* functor.(18)

In order to arrive at a categorical treatment of quantification, it was decisive that in a category $\mathbf{C}$ with pullback (fibred products), the contravariant functor $f^* : \mathbf{C}/Y \to \mathbf{C}/X$ associated with a morphism $f : X \to Y$ possesses both a left and a right adjoint. Moving from algebraic geometry to logic, the functor f^* in fact expresses the *substitution of* a term y defined on Y by a term x defined on X *along* f, determined precisely by $f(x) = y$. While the ordinary treatment of substitution found in logic textbooks omits f, its non-omission (facilitated by using variables of more than one sort) allows for a finer analysis, which fills a gap in previous attempts to extend algebraic logic from the monadic case to the polyadic (relational) case.

But what are the left and right adjoints of f^*? To fix ideas, it is best to restrict ourselves to the more familiar case, where X and Y are two sets and f is a function from X to Y. The functor

(17)Given a functor F from category $\mathbf{C}$ to category $\mathbf{D}$, and a functor G from $\mathbf{D}$ to $\mathbf{C}$, F is said to be left adjunct of G, and G right adjunct of F, if there exists a natural isomorphism between $Hom_{\mathbf{D}}(FA, B)$ and $Hom_{\mathbf{C}}(A, GB)$, for each object A in $\mathbf{C}$ and B in $\mathbf{D}$, where an isomorphism is said to be "natural" if it is stable with respect to any morphism $A' \to A$ and $B \to B'$. When this condition is satisfied, $F \dashv G$ written. Galois connections are a particular example of adjunction.

(18)For a more general formulation, see [23].

Sub associates to any given set the set of its subsets and, given a function $f : X \to Y$, behaves as its counter-image map, so *Sub* has as its domain the poset[(19)] of the subsets of Y and as its codomain the poset of the subsets of X. Usually we denote $Sub(X)$ as the power set $\mathscr{P}(X)$, which in the category of sets, **Set**, can be represented by 2^X. In this same category every function $f : X \to Y$ induces a functor $\mathscr{P}f : \mathscr{P}X \to \mathscr{P}Y$ s.t., if $A \subseteq X$ and $B \subseteq Y$ then $\exists_f(A) \subseteq B$ if and only if $A \subseteq f^{-1}(B)$ where $\exists_f(A) = Im(f) = \{y \in Y | \exists x \in A(fx = y)\}$; thus $\exists_f \dashv f^{-1}$.
But f^{-1} also has a right-hand adjoint, denoted $\forall_f$, because, given that $\forall_f(A) = \{y \in Y | \exists x \in X(fx = y \Rightarrow x \in A)\}$, it is true that $f^{-1}(B) \subseteq A$ if and only if $B \subset \forall_f(A)$; thus $f^{-1} \dashv \forall_f$. Generalising, the two adjoints of f^* correspond to $\exists_f$ and $\forall_f$ respectively, also in other suitable categories. If X and Y are spaces and f a continuous function from X to Y, *Sub* maps them to the collection of their respective opens, which does not give rise to a discrete topology, as $\mathscr{P}f$ would, and then, in a category of sheaves on a space, Ω does not necessarily have the structure of a Boolean algebra but that of a Heyting algebra, so the logical environment is no longer the classical but the intuitionistic one.

As far as logical connectives are concerned, let **P** be a category whose objects are propositions and whose morphisms are deductions, **1** be the terminal category having only one object, denoted by $\{*\}$, and, as the only endomorphism, obviously the identity, while Δ is the diagonal functor on **P**, which maps each proposition β to the pair $\langle \beta, \beta \rangle$. The product functor $- \times -$, from $\mathbf{P} \times \mathbf{P}$ to **P**, is right adjoint to Δ and induces the binary product with the properties of conjunction, while the coproduct (sum) functor $- + -$ as left adjoint to Δ lets **P** have coproducts, the properties of which correspond to those of disjunction. The two morphisms, $\bot$ and $\top$, defined as the left and right adjoints to the unique functor ! from **P** to **1** represent False (absurd) and True ($\top$ for *True*) respectively.

Finally, the right (exponential) adjoint to the product, with a fixed factor, gives rise to the implication $\Rightarrow$, by which the exponential $(-)^\beta$ is 'internalised' (i.e. represented internally). The negation $\neg\beta$ is a special case of this: $\beta \Rightarrow \bot$. If we make the further assumption that *by symmetry* the functor $- + \beta$ has a left adjunct, a new

(19) In the specific case where the objects are sets, *Sub* (-) gives rise to a complementary distributive lattice, endowed with a minimum and a maximum, so it is a Boolean algebra, whereas such is not the lattice of the opens of an arbitrary space.

connective $\setminus$, of 'logical subtraction', with properties different from negation, can be identified.

Intuitionistic propositional logic can then be expressed in a single diagram of adjunctions between functors, to which is added a second diagram of adjunctions that expresses the properties of quantifiers and is nothing more than a direct generalisation of $\exists_f \dashv f^{-1} \dashv \forall_f$ described above for sets, [32] and [36]. However, not everything that is expressible in a first-order (or higher-order) language is preserved along functors that have geometric meaning: it was therefore necessary to identify the subclass of formulae that are thus preserved.(20)

This concise description of how logical notions are presented in the categorical approach gives insight into the radical change in perspective that takes shape with Lawvere. The idea that one *could* categorically set up the *Grundlagenforschung* also entailed a change in *philosophical* perspective, similar to the change that took place when Galileo understood the state of rest of a body no longer as a *qualitative* state opposite to that of motion, but as a motion with zero velocity. The comparison is not accidental, because **Set**, the category (topos) of sets and functions between sets, is equivalent to the category of the sheaves of sets that vary on a point-like base space, i.e. to $\mathbf{Sh}(X)$ with $X = \{*\}$, on which the variation is zero. Id_X is obviously the terminal of this category, and $Sub(*)$ gives rise to classical logic. More generally, if X is the base space the structure of $Sub(X)$ is sufficient to determine the logic of a topos, at least as far as properties of a *local* character are concerned. Along the same path of modern science in mapping qualities to quantities that can be measured, also the difference, generally understood as qualitative, between extensional and intentional semantics takes a range of forms, dependent on the existence of a generator (or generating family) that may be quite different from the terminal. Consequently, it is possible to "measure" how much the procedural character of functions differs from their set-theoretic meaning as sets of *n-ples*, [34] for $n \geqslant 1$.

It was the setting of concepts and theories in a topos of sheaves that made it possible to successfully tackle intractable problems while remaining within the topos of constant sets, or in a topos of presheaves of variable sets on a set. To give just a couple of examples:

(20) After Lawvere's pioneering research, Gonzalo Reyes provided some of the main contributions on this issue, see [27].

- From 1967 onwards, Lawvere and Anders Kock gave shape to synthetic-differential geometry by setting it in a *smooth topos*, as a category whose objects are generalised smooth spaces (with smooth functions as morphisms) between which there is an object D that acts as an 'infinitesimal space' and all morphisms from D *to* $\mathbf{R}$ are linear; which is *only* possible in a non-Boolean context, [32].
- In 1982 Martin Hyland introduced **Eff**, the *effective topos*, as an appropriate category for realisability in Kleene's sense, because in **Eff** every total function from $\mathbf{N}$ to $\mathbf{N}$ is recursive, but again **Eff** is not Boolean, [19].

Although it must be acknowledged that from 1970 onwards the categorical approach to foundations developed independently of research in algebraic geometry, many concepts used in categorical logic(21) were derived 'by distillation' from their use in algebraic geometry. Among these concepts, some are to be considered key concepts:

1. the construction of sheaves of variable structures on a base space and the role assigned to 'local character' properties;
2. invariance by base change with respect to appropriate functors;
3. the notion of a *variable*, *generalised* element of an object X, as a morphism of codomain X, and the logical notion of a *constant* as a *point* (global section) of X;
4. adjunctions between functors as connective tissue between categories of different species (such as spaces, groups, sets) and the related process of internalisation (as described by Grothendieck in [14]) ;
5. the concept of a *generic* model of a ('geometric') theory.

Points (1) and (3) made it possible to elaborate a theory of continuously variable sets on a base space and then, by considering presheaves, a theory of variable sets in general on a category. Point (2) emphasised *geometric morphisms* between topoi, i.e. those pairs

(21) For an accurate reconstruction of the genesis of categorical logic, with particular reference to *geometric logic*, see [27].

of adjuncts between two topoi, when the left adjoint preserves limits (is 'left exact'). Point (4) prefigured the intrinsic link between logic and algebra, established through the existence of an object Ω. Point (5), within the framework of the functorial semantics introduced by Lawvere, contributed to the discovery of geometric logic, directly connected to point (2).

It is through these key-concepts that, in response to Enriques' initial question (*), the *philosophical meaning* of algebraic geometry takes shape, in a sense that differs from the ways in which the link between geometry and philosophy, and more generally between mathematics and philosophy, has been configured in the past (as will be seen in the next section). Logic and geometry are, in fact, no longer separable, once inferential principles are *determined* by the category in which they are 'internalised'. Whereas Brouwer's appeal to the intuition of time was associated with the demand for constructive proofs, which led to intuitionistic logic, now that logic is a direct consequence of the algebraic structure of Ω in a generalised space, of which temporal order is only one example; and whereas Brouwer's view was opposed to formalism, according to which proofs are reduced to manipulations, governed by logic, of meaningless symbols, now logical operations can be seen as projections, to 'syntactic' objects, of operations on objects of a universe U endowed with topological and geometrical properties, and this projection in turn feeds spatial intuition (in a manner analogous to how, after associating an affine variety with its ring of co-ordinates one passes to any ring and realises that it can be represented as a ring of functions on a scheme).

§ 5. — Remarks on the historical development of the relationships between geometry and philosophy.

The special nature of the entities referred to in mathematics and the special status of geometric knowledge have been the subject of philosophical reflection since antiquity. Concepts, principles and methods of mathematics have acquired philosophical significance for various reasons: first for the paradigmatic value attributed to mathematical entities, as in the case of the Pythagoreans' arithmo-geometry, and for the idea that mathematical rigour is the model of rationality; then for the indispensability of mathematics in the formulation of physical laws, as well as for the impact of certain mathematical results on classical philosophical questions, such as those concerning infinity, and finally for the philosophical interpretation of meta-mathematical theorems, such as the Incompleteness Theorem (Gödel) and the Theorem of Indefinibility of Truth (Tarski) when the expressive resources of language allow for self-reference.

Mathematics, and in particular geometry, was even understood as a prerequisite for doing philosophy, so much so that, apparently, the inscription Αγεωμέτρητος μηδεὶς εἰσίτω, 'Do not enter if you do not know geometry', stood out on the portal of the Platonic Academy. Today, a freshman philosophy student who, upon entering the building in which classes are held, would find such an inscription at the entrance, would be inclined to assume that he or she had got the wrong house number or would begin to doubt whether he or she had made the right choice. A mirror scenario for maths freshmen would have a similar effect.

To be generous, it is said to be just a practical necessity, resulting from the division of intellectual labour and increasing specialisation. If this had always been the case, however, today's science would not exist. To be less generous, it is said that mathematics is *now* one thing and philosophy another: a truism, but this may be misleading because it leads one to forget that 'disciplinary fields' do not come into the world with their boundaries written on their foreheads: the concepts and methods that distinguish them are the product of a long selection, and this selection, which at no point can be considered complete, is enriched as new connections are established between two areas of research.

Instrument (Ὄργανον) of knowledge: this is how the Aristotelians understood the theory of syllogisms, which today is only a (decidable) part of (classical) first-order logic. At the entrance to the Lyceum, however, Aristotle did not have an inscription similar to that of the Academy: 'Let no one enter who does not know logic'. In reality, just as the Academy was a place of training for many mathematicians (such as Eudoxus), so the Lyceum was a place of training for those who wanted to learn a method to ascertain the correctness of reasoning in any field of knowledge... assuming that *every* argument were reducible to syllogistic form, which geometry itself disproved (e.g., the transitive property of the identity relation does not correspond to a syllogism).

Logic was a philosopher's affair and did not require expertise in geometry and, more generally, mathematics was not considered by Aristotle to be essential to *natural philosophy* — which included what we now call 'physics'. The Aristotelian model of knowledge was taxonomic, and the characterisation of each individual entity was arrived at by means of a *qualitative* classification, the purpose of which was the actual (essential) definition of each node of the tree of genera and species, with the exception of the maximal genera and the lowest species.

Although the taxonomic ideal did not disappear, not even in mathematics — just think of the classification of surfaces, and the classification of simple groups — first the 'forgotten revolution' in the Alexandrian age, [41] and then the scientific revolution of the 17th century sanctioned a break with the Aristotelian approach even from a logical-linguistic point of view, passing from a qualitative to a quantitative discourse and from propositions of subject-predicate form to propositions that expressed relations between the increase/decrease of one quantity and the increase/decrease of another in a given time interval, and above all, at least as far as physics was concerned, the form of 'laws' became *equational.*

If in such a form we recognise a guiding idea of the scientific image of the world, we should be surprised that the principles of mathematics do not have the same form. On the contrary, many still take it for granted that the logical principles and those of set theory, on which all mathematics should be based, are not equations. Thanks to Lawvere, — and of course not only to him, — the extraction of logical principles from the very structure of a topos paved the way for removing this residue of the pre-Galilean view.

As everyone knows, equations require the symbol '=' to indicate equality. Equality is a very special relation because it is the finest *equivalence* that is also the most general congruence, i.e., that satisfies the substitutivity principle with respect to an algebra of properties (and operations). As a rule, in fact, the symbol '=' is used with reference to a given domain and a fixed set of assumptions. To give an example pertinent to algebraic geometry, the polynomial equation with integer coefficients $x^2 + y^2 - 3 = 0$ has no integer solutions, but if we go to the finite field of integers modulo 5, it has instead solution: $3^2 + 3^2 - 3 = 15$ (congruent to 0, precisely modulo 5).

In mathematical practice, equivalence relations are often used in view of the quotient structures associated with them. From a foundational perspective, they are required to reconstruct number systems by means of a chain of definitions by abstraction (of $\mathbb{C}$ from

$\mathbb{R}$, of $\mathbb{R}$ from $\mathbb{Q}$, of $\mathbb{Q}$ from $\mathbb{Z}$ and of $\mathbb{Z}$ from $\mathbb{N}$), so the principles that legitimise definitions by abstraction must be specified and this, in turn, requires the logical principles of reasoning using quantifiers and arbitrary relations to be made explicit. Both objectives were achieved in the late 19th century, after the arithmetisation (Dedekind, Cantor) of the Calculus, as the final step in the process of making mathematics the realm of rigour that it is expected to be.

In fact, in the same years in which Cantor laid the foundations of set theory, Frege laid the foundations of the new logic, in a language in which alternating sequences of quantifiers (for every x there is a y such that for every z ...) could finally be handled in the presence of *n-ary* relations, arriving at the definition of the cardinality of a set M as the equivalence class of all M' such that there exists a bijection between M and M'.[22] This bridged the gap between logic and mathematics that had lasted for more than twenty centuries, and which Boole had only minimally bridged by making the algebraic treatment of syllogistics possible. The logic needed to express mathematical reasoning also had an axiomatic form, just like geometry, but this form was not equational.

As for the axiomatic method as a model of rationality, if the Euclidean *Elements* have been understood, for two millennia, as something more than a treatise on geometry, it is because they exemplified this model, as the title of Spinoza's most famous work: *Ethica more geometrico demonstrata* testifies. The concept of proof, however, had remained *implicit*: used and exemplified, rather than made the subject of mathematical investigation. The clarification of its characters did not seem indispensable, even though the problem of the independence of the fifth Euclidean postulate from the other four could already signal its necessity. It was thanks to Hilbert and his school that the axiomatic method reached an extraordinary level of rigour, leading to *Proof Theory*, in which the very *structure of* proofs is studied.

And strictly speaking, axiomatisation typically has two faces: one side is turned towards *expansion*, because it induces one to explore the variants of the concepts usually employed as well as their possible generalisations — and in this respect the abstract algebraic approach proved to be a decisive example; the other side is turned towards the *foundation*, the paradigmatic example of which was the 'reductionist' perspective (from $\mathbb{C}$ to $\mathbb{N}$) developed during

[22] In categorical language, the corresponding quotient, like any other, is obtained as a co-equaliser.

the 19th century, which, however, stopped at arithmetic and arithmetic still lacked an axiomatisation. Where Kant was still of the opinion that arithmetic was only a set of intuitive rules of computation, Dedekind and Peano set themselves the task of axiomatising arithmetic and succeeded in doing so almost simultaneously. Then, within the Cantorian framework, the general notion of *number* (as cardinal, finite or infinite) was definable by abstraction.

At that point, the project of rigourisation could focus on the principles of set theory using the new logic, in order to provide a secure foundation for all mathematics. Since then, the philosophy of mathematics has experienced great growth, even if the attention philosophers have paid to mathematics has been directed, almost exclusively, not so much to the architecture of mathematics as to issues internal to the scenario opened up by the systems of Frege, Russell and Zermelo: How to avoid paradoxes? What infinitary commitment to accept? What meta-mathematical resources to rely on? What justification to give of 'challenging' axioms, such as the axiom of choice (in one of its many variants)? What lesson to draw from the limits encountered (Gödel) by Hilbert's Program? What rearrangement of the fundamentals is achievable by adhering to the demands of constructivity (which, at the very least, exclude recourse to proofs by contradiction)?

Faced with this scenario, when Enriques' words quoted at the beginning were written (1921), they had a *retro* flavour. Logicism, that is, the thesis of the reducibility of mathematics to logic, was not supported, nor was intuitionism, because the intuition to which Enriques referred was geometric. Cantor's 'paradise' extended far beyond the scope of such intuition and, above all, the 'experimental' approach of the Italian algebraic geometers did not align with the metatheoretical scruples of Hilbertian formalism. And when Enriques died (1946) the 'abstract' approach to algebraic geometry was beginning to take shape with the French school, in an environment close to Bourbaki's structuralism.

Bourbaki set aside the philosophical commitment required by logicism, formalism and intuitionism and instead used the axiomatic route indicated by Zermelo. In fact, if we overlook the tension between descriptive intent, which led Bourbaki to recognise three fundamental types of '*structures mères*', i.e. 'mother structures' (algebraic, order and topological), and normative intent (inherent in distinguishing 'good' compound structures from 'bad' ones), a specific version of set theory, comparable to ZF plus the axiom of choice, but without the axiom of foundation, [3] remained as the common basis of the treatises of the *Éléments de mathématique* (begun in 1939)[(23)]

Bourbaki wanted to give voice to the *working mathematician* alluded to in the title of a famous article of his from 1949, [2] in a spirit that was later well expressed by Jean Dieudonné when, with evident irony, he observed that mathematicians have more to think about than asking 'philosophical' questions: at best, they think about it on Sundays, in order to return on Mondays to their interrupted work. Hence, there was no need to fiddle with logical-linguistic scruples, to worry about the 'size' of the sets referred to when handling function spaces, or to adhere to constructiveness constraints.

The great advances made in algebraic geometry after the Second World War culminated in the treatise *Éléments de géométrie algébrique* (1960-1967), written by Grothendieck and Dieudonné. What made this progress possible was an increase in abstraction, independent of a foundational project, as well as a specific philosophy of mathematics and logical requirements. In contrast, the main form of 'scientific philosophy' that had taken shape in the Vienna Circle had made logicism an essential component of a program that understood philosophy as the activity of *clarifying language*, be it ordinary language or the formalised language of a scientific theory, and this clarification was to get philosophical problems out of the way, reducing them to errors of 'grammar', leaving the remaining problems to science.

It was a 'therapeutic' and at the same time 'prophylactic' task: conceptual hygiene passed through logical analysis and, since mathematics was reduced to logic, there was no need to concern oneself with questions pertaining to specific areas of mathematics, because it was sufficient to stick to the logical syntax of mathematical language and 'formal" semantics, in set-theoretic terms, as introduced in 1935 by Alfred Tarski (1933), later enriched in the form of 'possible words semantics'. In essence, the activity of clarifying the meaning of each proposition was *confined* to the use of entities in **Set**, while in algebraic geometry the key-concepts (1)-(5), listed in § 4, were developed, which go far beyond this category and,

[(23)] For details on this, see [24] and [28]. The main alternative to ZF was the NBG system, formulated by J. Von Neumann, P. Bernays and K. Gödel, which admits proper classes and adopts a peculiar version of the comprehension principle.

put to use by Lawvere, allow a finer, and mathematically meaningful, analysis of the syntax and semantics of a formal language.(24) Not only: when the semantics is referred to a category of variable structures and, in particular, of variable sets, one regains (as already anticipated) the 'possible worlds semantics' for modal logic and intuitionistic logic, since such semantics basically has to do with a topos of presheaves of sets that vary on a simple poset (and if the poset has only one element, one obtains Tarskian semantics). Here again the two sides, *foundational* and *expansive*, of abstraction find joint manifestation.

In the same years Enriques was advocating a scientific philosophy unrelated to the need of logical rigour, that the only (reliable) philosophy is that which begins with the *analysis* of *language* was an idea made possible by the emergence of mathematical logic, whose use in such analysis was not, however, mediated by the topological, geometric, algebraic structure of the universe-of-discourse. The resulting confinement to set-theoretic semantics has favoured the flowering of a new scholasticism. The enormous gap between the 'purely' logical plane and the specific structure of each specific universe-of-discourse ended up being filled with exercises in pre-Galilean ontology, which have nothing to do with the mathematical language of modern science. On the one hand, philosophy was vowing itself to a *metalinguistic exile*, on the other hand, the exile was prevented by an object-language capable of self-reference. Now, in the name of rigour, bridges were cut with intuition, and now, in order to recover intuition, rigour was set aside. In short, if philosophy is reduced to an activity of clarifying language, it provides no knowledge at all; otherwise, it returns to host, at best, a metaphysics, 'analytically' moulded, which, however, lacking a link with mathematics as the *organon* of the explanation of the physical world, takes us back to a qualitative vision, setting aside any relationship between quality and quantity.(25)

Meanwhile, if the universe of sets led some to attribute to it a reality *parallel to* physical reality, the spread of the formalist attitude led others to think of mathematics as nothing more than a

(24) And later also of a natural language: see, in this regard, some of the contributions collected in [26].

(25) As if the link between the shape of a common doughnut and the $\mathbb{Z}\times\mathbb{Z}$ group had nothing to do with the concept of *path*, which has its root in the common experience of space, whereas the intuition behind such a concept is 'lifted' to the concept of proof.

collection of symbol manipulation games.[26] The neo-positivists had got around the alternative: since mathematics is reducible to logic via appropriate definitions, mathematical truths are such in virtue of the meaning of the terms alone, hence they are 'analytic' truths and, as such, inherent exclusively to language, and since the choice of one language over another is conventional, mathematical assertions have no content until they receive it from their use in the empirical sciences. Behind this idea was Poincaré's solution to the question that emerged with the discovery of non-Euclidean geometries: Which geometry is *true*? Poincaré argued that the question was ill posed: one should rather ask what metric convention to adopt and, as with any other convention, the choice being free, it is only a matter of assessing which one is more effective, [39].[27]

But even independently of logicism, one could avoid any reference to intuition by understanding the axioms of a mathematical theory as implicit definitions of primitive notions; consequently, the (formal) meaning of geometrical notions was univocally fixed by the network of inferential links within an axiomatic system. This idea, put forward by Poincaré, in polemic with Russell, even before Hilbert, is still widespread today, despite the fact that for any first-order theory that has an infinite model, that model cannot be unique (up to isomorphism). But even setting aside logical questions concerning the idea of a system of axioms as an implicit definition, such an epistemological framework fails to account for the progress made in the twentieth century by the mathematical investigation of the concept of space, and prevents one from grasping the *philosophical* relevance of its many aspects. It suffices here to recall four points:

I. the paradigmatic character of the method of local maps and their gluing, as elaborated in differential geometry;[28]

[26] One can, therefore, well understand that the effectiveness of mathematics in the natural sciences should appear 'unreasonable',[45].

[27] Within the neo-positivistic framework, this way of understanding geometry expanded into a general conventionalism, not only concerning the adoption of one axiomatic system of geometry rather than another, but also the adoption of the principles of any physical theory. In doing so, the neo-positivists set aside Poincaré's own recognition of the role of intuition in mathematics. Poincaré, in fact, regarded both the axioms defining a group and arithmetic induction as non-analytic but synthetic a priori principles.

[28] Contrary to the neo-positivist lesson, the philosophical scope of this method cannot be confined — as it was not confined in Hermann Weyl's monograph

II. the presence of topological concepts in every area of mathematics;

III. the change in perspective due to 'abstract algebra', which found systematic formulation in Bartel van der Waerden's *Moderne Algebra* (1930-31);

IV. and of course the specific advances in the study of spaces the points of which are roots of polynomial equations, as in algebraic geometry.

It is rare to find a trace of these points in the philosophy of 20th century mathematics, almost exclusively focused on logic and set theory, as well as in the philosophy of science. Which is curious, to say the least, because

I*. Riemann had been a student of a philosopher such as Herbart, heir to Kant's chair at Königsberg, and it was from Herbart that he took his cue to focus on the notion of extended manifold (*Mannigfaltigkeit*);

II*. *analysis situs*, from Poincaré onwards, has a direct impact on Kant's statements concerning space *that did not refer to metrics*;

III*. the group concept is ubiquitous in mathematics, as it is in physics;

IV*. the definability of the entities referred to is a classical theme in philosophy, so when the identification of something is a function of the complexity of its definition, algebraic equations should have been the starting point.

In the philosophy of language, the expressibility of a principle in equational form did not carry the weight it deserved, and when, in the philosophy of science, it came to be hypothesised that logical principles could depend on physical-geometric structure,(29) the lesson was that of empiricism, opposed to the Riemannian idea of an *intrinsic* characterisation of the properties of a structure: the analogous possibility of deriving logical properties *from within* a

on Riemann surfaces (1913) — to Einstein's later use of it in general relativity, understanding space-time as a Riemannian manifold. For an example, relevant to Willard v. O. Quine's criticism of the two 'dogmas' of empiricism, see [35].

(29) The main impetus for this came from quantum mechanics. Whereas the set of events of a classical system can be described as a Boolean algebra, the set of events of a quantum system is not a distributive but an orthomodular lattice.

sufficiently structure-rich U was excluded by the conventional character of the choice of a system of logic. Instead, this possibility took shape, thanks to Lawvere, in a fruitful manner and was articulated with the development of the five key concepts that emerged in algebraic geometry, as mentioned in § 4. The idea of a *conceptual mathematics* has thus found expression in the centrality assigned to the adjunction between functors, in such a way that what has *foundational* value is that which is transversal to the areas of mathematics, [20] and so attention cannot but shift to the universal constructions that allow the areas — and, with them, the multiple sources of intuition — to be connected.

§ 6. — What is meant by 'philosophical significance'?

The 'philosophical significance' of ideas developed in any field of knowledge is a terribly vague expression, which risks condemning its analysis to superficiality. To reduce the vagueness and risk, it is necessary to refer to specific concepts, methods and principles that have found expression in a given field, to analyse the way in which they have been specified, and to focus on the change they have undergone in the course of the development of research in that field. By examining, for the sake of brevity, only the change, it should be considered that its philosophical significance may be (A) explicit or implicit and (B) direct or indirect.
With reference to algebraic geometry:

(#) As mentioned in the previous sections, the philosophical significance of the change has, for a long time, remained *implicit*, because, even restricting it to the key-concepts (1)-(5), behind them there was no intent similar to that of Leibniz, when he placed the concepts of the Calculus within the project of a *characteristica universalis* of combinatorial nature, or similar to that of Frege, when he developed an ideography (*Begriffsschrift*) in view of a logical foundation of mathematics; on the other hand, neither the Hilbertian school's demand for rigour nor the need for increasing abstraction would have been sufficient to identify the key concepts. It was only with Lawvere that the general scope of the key concepts was made explicit.

(##) While an *indirect* philosophical significance can easily be traced in any relevant advancement of mathematical knowledge, thus also in the development of algebraic geometry, its specific, *direct* meaning is not found in Enriques' observations on mathematics, nor in those of Grothendieck: for example, the method metaphorically expressed by the immersion of the walnut in water and the *rising sea* lent itself rather to being placed in the framework of structuralism, but an elaboration in this sense did not take place.

The elucidation of an implicit and indirect philosophical significance takes time. This might suggest the Hegelian idea of philosophy as a nightingale that takes flight at dusk, i.e., after the fact, were it not that in the present case such an idea would be erroneous, because philosophical motives are not absent from the path of mathematical research: in particular, the very project of a *conceptual mathematics* called into question the assumption that the only channel of interaction between mathematics and philosophy is through the essentially meta-mathematical questions *about* first principles, primitive notions, methods of proof and types of definition. This assumption supports the idea expressed by Dieudonné that the everyday work of mathematicians does not require any philosophical engagement, this engagement remaining confined to frame questions, such as: what is space? What is a number? What distinguishes mathematical knowledge from other types of knowledge? What meaning is to be given to mathematical truth? And the answers to such questions do not belong to mathematics and have minimal if any repercussions within it.

Such an idea seems to ignore the fact that in the 20th century these questions were addressed *by mathematical methods.* In the case of algebraic geometry, its philosophical significance emerged through a deepening of the concepts used in mathematical practice, aimed at solving specific problems unrelated to foundational questions. Decisive in this solution was the use of new concepts that placed the *architecture of* mathematics at the centre, but in a sense not imagined by Bourbaki. Rather, these concepts freed Bourbaki's conception of *structures mères* from a pyramidal residue (with the apex of the pyramid in set theory) and made architecture properly modular, at the same time making explicit the systematic interweaving of categories of different kinds, each anchored in basic experiences that nourish intuition.(30) It is in this second perspective that, through category theory, the implicit and indirect philosophical significance of algebraic geometry has become, with

(30) Mac Lane himself arrived, in [25], at such an idea, which, rather than opening up to a more refined (categorical) structuralism, can be understood as a 'genetic' phenomenology of mathematics, [38].

Lawvere, direct and explicit: the key-concepts refer, in fact, to constructions that are *transversal* to the areas of mathematics and find expression in pairs of adjoint functors, [19].

In 1971, Mac Lane also stated: 'Adjoints are everywhere'. For Lawvere their ubiquity did not lead to a version, revised and corrected in categorical terms, of structuralism, but rather to the recovery of dialectical materialism. Lawvere's endeavour in this regard seemed vague to many mathematicians and, because it by-passed the current debate in the philosophy of mathematics, prevented philosophers from understanding the revolutionary scope of the new conceptual framework for a long time. In fact, what was missing was an epistemological step, namely, the recognition that the aforementioned *transversality* is still described by means of the kinaesthetic patterns that shape the ordinary experience of the world, which is, fundamentally, the experience of the curves of moving bodies and their interacting surfaces, so that the battery of *patterns* that forms the 'base space' of cognition is *lifted* to the multiple planes of mathematical abstraction (including logic).

Such a change in perspective has important repercussions on the way we approach numerous questions concerning semantics and epistemology. The fact that, to date, these repercussions have received little attention testifies to the difficulty in breaking out of the 'analytic' paradigm, which nevertheless made the greatest contribution to 20th century philosophy.

At a time when the tools of mathematical logic are indispensable to philosophy and semantics acquires centrality, one cannot ignore the scope of the key concepts, which becomes decisive with regard to two problems, which, picking up the Einsteinian lesson, are also configured for mathematics:

(α) how to guarantee the invariance of truth from one reference system (universe-of-discourse) to another,

(β) how to guarantee that the transition from one logic to another has no effect on the theorems proved.

First, if a reference system is intended to be a topos, anyone who takes up Grothendieck's invitation to identify topos-independent principles will find themselves investigating the conditions that make *mathematical relativity* possible, and thus searching a solution to problem (α). Second, the question raised by (β) is not from which logic to start, but how to guarantee the stability of results with respect to a change in logic.

As for the problem (α), its solution, i.e. the identification of which properties, expressible in categorical language, are invariant for which transformations between one topos and another, is linked to the complexity of the formulae with which a given property P of a topos **E** is expressed. *Geometric logic* is precisely that which, in sequential form (*à la* Gentzen), governs the proofs that are limited to the use of such formulae. The fact that this logic is called "geometric" has to do with *geometric morphisms* between a topos **E** and a topos **E'**, where a geometric morphism between **E** and **E'** is given by a pair of functors $f_* : \mathbf{E} \to \mathbf{E}'$ and $f^* : \mathbf{E}' \to \mathbf{E}$ such that $f^* \dashv f_*$ *and* f^* is left exact (i.e., f^* preserves finite limits and arbitrary colimits). In fact, for any property P, if P is expressed by a geometric formula that is valid in **E'** then the component f^* of such a geometric morphism preserves P, hence the property is transported to **E**. However, the transport is not ensured in the reverse direction, from **E** to **E'** — which prevents a perfect analogy with the idea of relativity in physics, since Lorentz transformations form a group, hence are invertible.

In algebraic geometry, the concept of a *coherent topos* was introduced as a topos of sheaves on a site that admits a Grothendieck topology with characteristics similar to those used to define the compactness of a space. A *point of* a topos **E** is a geometric morphism $p_* : \mathbf{Set} \to \mathbf{E}$ with $p^* \dashv p_*$. A topos has *enough points* if for any two morphisms $f, g : A \to B$ in **E**, $f \neq g$ implies $p^*f \neq p^*g$. Pierre Deligne, who worked with Grothendieck, proved a theorem (contained in SGA 4), referred to as 'Deligne's Theorem', which states that *every coherent topos has enough points.*(31) This theorem was recognised by André Joyal as a version of the Completeness Theorem for first-order logic.

As for problem (β), its solution is ensured, in the case of the change from intuitionistic to classical logic. What is referred to as 'Barr's Theorem', due to Michael Barr, states that if a geometric formula $\varphi(x_1, \ldots, x_n)$ can be deduced using classical logic from a theory T, where the axioms of T are also geometric formulae, then $\varphi(x_1, \ldots, x_n)$ holds in every model of T in every Grothendieck topos.

(31) A topos is coherent if it is a topos of sheaves on a site in which every covering of an object is finite (which logically corresponds to the fact that finite disjunctions are sufficient). For an explanation of Deligne's Theorem and, more generally, for a step-by-step introduction to the relations between topos theory and geometric logic, see [13], ch. 16.

The solution to problem (α) emphasises the role of geometric logic, which is not a mere subsystem of ordinary intuitionistic logic, because infinitary disjunctions are admitted. The solution to problem (β) illustrates that under appropriate conditions the difference between intuitionistic and classical reasoning does not only disappear in the case of finite models (as Brouwer had already admitted) but also in the case of models in a topos. The proof of Barr's Theorem, however, is not constructive, and if we move on to universes of discourse that are not topoi, other logics are encountered and so both problem (α) and problem (β) arise again. To tackle them, the methods distilled from algebraic geometry may not suffice, but that does not detract from the fact that it was those methods that allowed Lawvere to connect three far-reaching Theses:

(T1) a foundational perspective is possible which, instead of starting from constructive or computational motivations to adopt a logic, *intrinsically* determines the logic as a function of a sufficiently structured mathematical universe-of-discourse (such as, for example, a topos);

(T2) one can disregard $\in$ to characterise the structure of a mathematical universe-of-discourse;

(T3) founding is not the same as finding axioms that identify an ambient mega-category as a sort of 'absolute space'.

§ 7. — Concluding remarks.

Theses (T1)-(T3) are met with various objections and have in fact been contested by set-theorists, logicians and philosophers. Against (T1): Even assuming that ideas from algebraic geometry were useful for 'categorical logic', the importance of those ideas is reduced to their heuristic value. Against (T2): autonomy from a set-theoretic meta-theory has not been proven. Against (T3): pluralism with regard to mathematical universes-of-discourse tacitly presupposes a unitary framework, which after all motivated the very investigation of **Cat**, the category of all (small) categories. A common element to these objections is the denunciation of a circularity flaw. These three objections can be variously answered. To give just one example about (T1), the identification of which logics correspond to which types of categories is an essential refinement of classical model theory, which remains a pillar of logic.(32)

Bourbaki *might have* looked favourably on these three Theses were it not for the fact that, as mentioned above, the first book of the *Éléments de mathématique* relied on an axiomatisation in terms of $\in$, so when it came to the notion of a set, the structuralist approach was put on hold and the *received view* was aligned. But even when one makes the hierarchy of sets depend on the complexity of the formulae defining them or on the constructive character of proofs (see intuitionistic set theory), one assumes that the logical structure is autonomous from the structure of the universe-of-discourse; and since there is a primary and all-encompassing universe-of-discourse, the problem of invariance does not arise.

On the one hand, the logic used in describing *local* properties (relative to an ambient set) of the structure of the universe of sets is intended to coincide with that used to describe its *global* properties and also with that used in the meta-theory. On the other hand, the plurality of logics can hardly be reconciled with an 'absolute' system of reference, and so we fall back on reasons of mere utility behind which an extreme relativism thrives, perfectly in line with the character of *linguistic conventions* attributed to logical principles by Rudolf Carnap as early as 1934 (§ 17): "our attitude is expressed through the formulation of the 'principle of tolerance': it is not our task to establish prohibitions, but only to arrive at conventions [...]. In logic there are no morals. Everyone is free to construct his own logic, i.e. his own form of language, in the way he wants', [4].

But if each universe-of-discourse has its own logic, what is the logic of the meta-language in which it is recognised that the logic of a given universe-of-discourse U is *different* from that of another U^*? Could the logic used to justify conventionalism be considered as conventional as that relating to U and that relating to U^*? Or was the Riemannian demand for intrinsicity only confined to geometry?

As we have seen in § 6, it is possible to establish the invariance of the truth of geometrical propositions from one topos to another under precise conditions. Just as *it was not obvious* that there could be an intrinsic geometry, so *it is not obvious* that there can be an

(32) For an extensive historical reconstruction of the origins and subsequent development of categorical logic, starting with Lawvere's insights, but especially thanks to the research of André Joyal and Gonzalo Reyes, see [27], with specific reference to categorical model theory.

intrinsic logic and that there is a close connection, far from conventional, between the invariance of the truth of a proposition (from a model in one topos to a model in another topos) and that particular type of adjoint functors between one topos and another, which are called 'geometric morphisms' not by chance. But if problems (α) and (β) have philosophical significance, so does their solution.

It cannot be denied that set theory has a broader scope than algebraic geometry and that, in fact, concepts and principles of logic and set theory are employed in algebraic geometry. This, however, does not close the matter because the very search for invariants presupposes access to a battery of notions whose content is rooted in underlying intuitive patterns of *spatiality*. The gap that Enriques did not fill has only begun to narrow thanks to the developments in algebraic geometry and the associated key concepts in categorical terms, whereas the gap has remained unchanged in the main foundational approaches:

- Frege, the father of logicism, after eliminating all reference to intuition from arithmetic, came to think that intuition is essential in geometry, so we should admit that there are mathematical propositions whose truth is, in Kantian terms, synthetic a priori after saying that mathematics is reducible to logic and that logical truths are analytic.

- Brouwer referred to an intuition that is linked to the notion of time in order to exclude non-constructive reasoning, but when we speak of the structure of time, we (metaphorically) exploit spatial notions, and the topologist Brouwer did not prove his fixed point theorem by adhering to constructive scruples.

- Hilbert confined the reference to intuition on the level of (finite) symbol manipulations, but what makes this intuition possible after setting aside geometric intuition was not explained.

As for the controversy between Hilbert and Frege over the notion of truth in mathematics, the lesson drawn in the twentieth century was that Hilbert was *obviously* right and Frege was *obviously* wrong, because the only thing that matters is to ensure that a set of axioms is non-contradictory; moreover, semantics is reduced to the translation of a theory T into the syntax of set theory (supposedly non-contradictory).

Category theory incorporates the idea that, in addition to structures, individual objects satisfying a property expressible in the language of the theory are also only identifiable up to isomorphism, so one might have the impression of a further departure from the intuitive notions of 'meaning' and 'truth', were it not for the fact that the picture of semantics changes the moment theories themselves are treated as categories, so that their models are the image of functors, thereby constraining the range of interpretations, and at the same time the class of models expands because it also includes models in categories other than sets. No less important is the fact that if a theory, understood as a category, has a *generic model*, then every other model of it is derived from that (via the unit of an adjunction), and that the methods required for this, when considering theories expressible in a 'geometric' language, were elaborated within the framework of algebraic geometry. But the ability to understand a commutative diagram still has to do with an intuition that is presupposed by the theory and involves kinaesthetic patterns that in turn require a mathematical description.

In recent decades, the problem of foundations seems no longer to be felt as it was until the 1970s, not so much because the sense given by Lawvere to the foundational task has been generally accepted, but for two basic reasons and one, so to speak, accompanying reason.

The first reason is that, with the explosion of so many new areas of research, and in particular those related to computer science, a pragmatic mentality has spread that leads one to use mathematical tools useful for one's research purposes, without bothering about the foundations on which these tools rest. The second reason is that foundations, ever since they were established as an object of mathematical investigation, have become an increasingly specialised area with an increasing number of ramifications, until they have become a labyrinth far removed from actual mathematical practice. The accompanying reason is that whereas the philosophical positions referred to used to be one with a research program in mathematics, for some time now research in the philosophy of mathematics has been predominantly oriented towards comparative analysis and looks at the work of mathematicians from the outside without direct collaboration with them.

However, in recent decades, the landscape of the relationship between logic and category theory has also been enriched by new lines of research, which, starting from the identification of the 'internal' logic of a topos, have focused on the language of type theory.

Categories without diagonal maps were identified, thanks to which it was possible to formulate the categorical semantics of linear logic and *locally* closed Cartesian categories. For such a category **C**, the *slice* category **C**/ A over any object A is Cartesian closed, without **C** having the terminal: these categories were used to interpret theories admitting types dependent on the terms of other types.

Then there are the categories that have provided models for second-order lambda-calculus, in which what in theoretical computer science is called 'polymorphism' (for types-of-data and functions between them) is expressed. Indeed, the interest in higher-order functional programming languages has led to categories that require additional resources to those expressible in a topos and, thanks to such resources, correspond to constructive theories of types (such as that of Per Martin-Löf) of particular relevance to theoretical computer science, up to the formulation of a *homotopic* theory of types (developed from an idea of Vladimir Voevodsky), proposed as a foundation for all mathematics.

The fact remains that it was the growing abstraction that took shape in algebraic geometry that highlighted the key concepts that then allowed a close relationship between logic and geometry to be specified. This relationship, in turn, led to the identification of 'universal' constructions, configuring a new sense to be given to the foundations and at the same time setting up the tools to arrive at a satisfactory answer to the question (*).

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Alberto Peruzzi,
Università degli Studi di Firenze