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The significance of criticism of principles in the development of mathematics

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Summary: I: Introduction. II: The continuum and infinitesimal procedures in antiquity. III: The foundations of infinitesimal calculus. IV: The criticism of infinitesimal concepts and new developments in the calculus of variations. V: Arbitrary functions and the modern elaboration of the concept of the continuum. VI: The intensive development of mathematics: equations and imaginary numbers. VII: Riemann's theory of algebraic functions and the criticism of the principles of geometry. VIII: New developments in algebra. IX: Conclusions; pragmatism and mathematical naturalism. X: Mathematics as a tool and model of science.

(1) Translation by F. Patras of: Federigo Enriques, Il significato della critica dei principi nello sviluppo delle matematiche, *Scientia* XII (1912), pp. 172-191. The original article has been republished in the *Proceedings of the Fifth International Congress of Mathematicians* (Cambridge, 22-28 Aug. 1912), Hobson, E. W., Love, E. H. eds., University Press, Cambridge, 1913, pp. 67-79 and, more recently, in *Per la scienza*, Simili, R. ed., Bibliopolis, Napoli, 2000. A French translation was published in the Supplement to the same 1912 issue of *Scientia*, pp. 59-79.

(2) [NdT] I would like to express my sincere thanks to Paola Cantù and Frédéric Jaëck for their help and comments on a preliminary version of the translation.

§ 1. — Introduction.

Criticism of principles is on the agenda among contemporary mathematicians. In-depth analysis of the concepts of limit and function, research based on the theory of parallels and non-Euclidean geometry, more recent research linked to the foundations of projective geometry and *Analysis situs*, developments concerning multidimensional varieties, transformations and their groups; finally, set theory and speculations on the infinite and the actual infinitesimal, to which non-Archimedean geometries are connected, have raised many problems that touch the deep roots of the mathematical edifice and attract, for various reasons, philosophical minds.

In the context of an eminently conservative science, which for two thousand years has offered the spectacle of uninterrupted continuity of progressive constructions without demolition, innovative, revolutionary criticism may elicit a stronger emotional response than in any other field of knowledge, where crises occur visibly on a periodic basis. This emotional interest is responsible not only for the resistance that new ideas encounter among a public unprepared to understand them, but even more so for the seduction they exert on so many minds, ready to move, by natural psychological reaction, from wonder and bewilderment to faith and enthusiasm for the new world that is opening up before their eyes.

Hence the singular phenomenon we have witnessed many times: the propagation of critical ideas through small circles of workers and interpreters who, developing their logical consequences to the extreme, carry out a veritable apostolate around themselves, perhaps fooled by the illusion that the new truth they have discovered will mark a radical revolution in mathematical thought and usher in a new era in its history.

We must thank the multiplicity of Churches if the fervent propaganda around us does not take away our sense of relativity and allows us to retain some faith even in old mathematics.

Now, the liveliest discussions aroused by the new fields of investigation and, above all, the new attitudes of the critical spirit naturally pose a philosophical and historical problem: what is the true value of the criticism of principles and what place does it occupy in the progress of our science? All the specific questions of evaluation, with regard to different approaches to analysis and research, seem to be dominated by that general problem which, albeit in different ways, every worker, reflecting on her or his own work, is led to ask herself or himself.

§ 2. — The continuum and infinitesimal procedures in antiquity.

History offers us an instructive first lesson in this regard: the criticism of principles is by no means a new phenomenon characterising the mathematical production of our times; on the contrary, it is an essential part of the elaboration of concepts that in every age prepares or accompanies the progress of science and its most extensive application.

The universally admired perfection of Euclid's work reveals itself to the historian as the mature fruit of a long period of criticism, which took place during the constructive period of rational geometry from Pythagoras to Eudoxus. Such subtlety and depth of ideas unfolded in that critical movement that certain views could not be understood until very recently, when the developments of our own criticism led us to truly surpass Greek thought in this direction as well.

Then, in particular, the significance of the methods and principles by which the Greeks themselves managed to overcome the paradoxes that seem to arise naturally when reflecting on the infinite began to become clear in its own light, as the difficulties that long troubled the mathematicians and philosophers of antiquity in this regard are the same as those experienced by the Renaissance in the constructive period of infinitesimal analysis, and even after its establishment, up to the most recent criticism.

The foundation of a theory of measurement by the Pythagorean school raised the question of the geometric continuum for the first time. The Pythagoreans based that theory on an indivisible element of space, the point with finite extension; meanwhile, the incommensurable ratio of the diagonal to the side of the square raised an insurmountable contradiction in their steps.

However, P. Tannery showed that only the criticism of the Eleatics succeeded in definitively overcoming the erroneous concept of the Pythagoreans. As is generally the case with certain abstract constructions, the paradoxical negative aspect of Zeno's arguments (Achilles and the tortoise!) must have struck the imagination of the general public, and this impression has passed into the current of literary tradition, where it is still dominant. But the positive value of this criticism is that it paved the way for an accurate view of the continuum and a theory of incommensurable quantities.

This theory was founded by Eudoxus of Cnidus through the introduction of the postulate, commonly referred to by the name of Archimedes, which serves as the basis for the general treatment of proportions, set out in Book V of Euclid.

Eudoxus' criticism also made it possible to provide a rigorous basis for the infinitesimal procedures used by the ancients to measure areas and volumes. In fact, Eudoxus himself based the process of exhaustion on his postulate and used it to demonstrate the results on the volumes of the pyramid and the cone, already found by Democritus.

The mathematical public, thirsty for rigour, applauded Eudoxus' work, and Archimedes' testimony (in the work rediscovered by Heiberg) tells us that, precisely in homage to rigour, it was not permissible to cite any other author of such doctrines other than the one who had succeeded in removing all objections by establishing the result with impeccable logic. Zeuthen wittily notes that it would be just as valid to attribute the discovery of infinitesimal calculus to Cauchy, who gave the final answer to the doubts raised by the use of infinitesimals!

The method of exhaustion was therefore the term at which the development of infinitesimal procedures consciously came to a halt among the Greeks, but the concepts underlying it, which were deliberately banned for the sake of rigour, come to light at every turn in Archimedes' work. And the letter he wrote to Eratosthenes reveals how the methods of infinitesimal analysis, the reduction of the continuum to a sum of a finite number of terms, served as a guide in his discovery, while the presentation of the results, conducted using the method of exhaustion, allowed him to satisfy the demands of the scientific public.

§ 3. — The foundations of infinitesimal calculus.

Archimedes' ideas were taken up and developed further during the Renaissance by Galileo and Kepler, to whom is associated the first organic arrangement of these ideas, which is Bonaventura Cavalieri's geometry of indivisibles.

The great Italian geometer posited the fruitful principle that surfaces and volumes can be regarded as sums of an infinite number of indivisible elements, which are respectively lines or surfaces, and drew very general and important conclusions from this. Attacked by Guldino in 1640, he showed that his method amounts to the exhaustion of the ancients; it was nothing more than a *fiction*, useful for the rapid solution of problems, and did not involve any *hypothesis* contrary to the traditional concept of the continuum.

Meanwhile, infinitesimal methods came to light in various forms; the fundamental difficulty in grasping their true philosophical meaning led to divergences in these attempts, which went hand in hand with positive achievements.

Torricelli and Roberval obtained the tangent by composing movements; and these decompose surfaces and solids into an indefinite multiplicity of rectangles or prisms decreasing according to a certain law.

The new procedures investigated by Cavalieri, Fermat, Descartes, and Roberval were further developed in Wallis's *Arithmetic of Infinitesimals*, from which results the first example of the rectification of a curve, and then by Mercator, who determines the area between the hyperbola and its asymptotes by means of a series. Finally, Barrow, Newton's teacher, who may be especially associated to Galileo and Torricelli, highlighted the inverse nature of the operations involved in determining the area and tangent of a curve.

It remained to be discovered that the latter is a *direct* operation that can be easily performed. This led to the organic constitution of modern infinitesimal analysis, i.e. Newton's method of fluxions and fluents and Leibniz's differential and integral calculus.

A great conceptual development, rooted in the most ancient Greek thought, thus presided over the acquisition that will remain a title of honour for the human spirit: the concepts of infinite and of potential infinitesimal [*infinitesimo potenziale*], which, later freed from all obscurity, will become the solid basis of calculus, represent, so to speak, the synthesis of the Pythagorean hypothesis, taken up as fiction by Cavalieri, and Zeno's negative criticism, converted into a rigorous process of demonstration thanks to Eudoxus' postulate. The synthesis will become logically perfect when Cauchy will succeed in reconciling the differences of opinion that continued to separate Newtonians and Leibnizians for a long time, as we shall see later. In the meantime, in order to appreciate the full extent of the analytical work accomplished, it is necessary to bear in mind the elaboration of the principles of mechanics that accompanies it. The same fundamental idea that constitutes the transition from the finite to the infinite and from the discrete to the continuous determines the general design of modern science, that is, the principle of a universal determinism that breaks down natural processes into a continuous series of *elementary causes*, and thus finds in the form of differential equations the invariants that constitute the object of a rational representation of reality. From this point of view, the rationalistic metaphysics of the schools of Descartes and Leibniz appears as a grandiose branch of the same criticism of principles from which infinitesimal calculus emerged.

§ 4. — The criticism of infinitesimal concepts and new developments in the calculus of variations.

I have said that, after Newton and Leibniz, criticism continued for a long time with the aim of giving a logical basis to infinitesimal analysis, whose fruitfulness appeared more and more marvelous every day.

Newton's method, which introduces fluxions as velocities, was the first to achieve a rigorous structure thanks to the criticism of Maclaurin and D'Alembert; by eliminating the dynamic concept in order to develop purely analytical principles, they recognised its logical foundation in the theory of limits. Newton's foundation thus allows for the ordinary calculation of derivatives. However, the speed allowed by the use of infinitesimals, to which Leibniz's more generally adopted notations conform, still called for a full justification of the fundamental hypothesis encountered in this way, namely Leibniz's principle that "infinitesimals can be neglected in the face of finite quantities and infinitesimals of a higher order in the face of those of a lower order".

The difficulty of logically understanding this principle, which seems to break with the mathematical spirit of exactness, still troubled Lagrange, who prompted the Berlin Academy to hold a competition, in 1784, with the aim of obtaining a rigorous treatment of infinitesimal analysis. A prize was then awarded to Lhuilier's memoir, which essentially sought to eliminate the fruitful characteristic of Leibniz's calculus.

There was a fear that logical scruples would once again, as in the ancient world, take precedence over the fruitfulness of the methods. But these now had too broad a basis in the mature awareness of the progress that had been made. A movement soon arose in reaction, producing S. Carnot's *Réflexions sur la métaphysique du calcul infinitésimal* which contains the idea that Leibniz's principle finds its justification, as a practical rule, in the consideration of the arbitrariness of the infinitesimal. Later, Cauchy demonstrated that this consideration, independently from other less clear reflections by Carnot, is sufficient in itself to legitimise Leibniz's calculus and to establish its identity with the theory of limits.

If we now wish to properly appreciate the value of the new criticism, which succeeds in logically establishing the foundations of infinitesimal calculus, we must reflect on the intimate connection that links this criticism to other positive developments in the minds of the aforementioned mathematicians, remembering that it was precisely from the attempt to rigorously define the principles of calculus that Lagrange was led to introduce analytical functions, which were systematically treated by Cauchy, Riemann and Weierstrass.

It is also worth mentioning the link between the criticism of principles and another great discovery by Lagrange, the calculus of variations, which is precisely an extension of infinitesimal concepts to the study of functions dependent on other functions and their maxima and minima.

In a series of lectures recently given in Paris, Vito Volterra(3) illustrated the progress of the theory of these functions of lines, showing how the evolution of the fundamental ideas of infinitesimal calculus continues here in the flourishing construction of the doctrine of integral and integro-differential equations; so that this admirable extension of concepts reveals itself as the ripe fruit of that same criticism that we recognised as the active ferment of progress in mathematics during two millennia of history.

§ 5. — Arbitrary functions and the modern elaboration of the concept of the continuum.

The field of criticism in the nineteenth century was broadened by the extension of the concept of function, gained in part through the problem of vibrating strings and the studies of D'Alembert and Fourier, and in part through the qualitative consideration of integration algorithms.

The first path suggests Dirichlet's concept of the arbitrary function, while the second, with Abel and Jacobi, leads to the effective introduction of more general functions into analysis.

(3) See *Revue du Mois*, 10 mars 1912.

It is precisely the consideration of these, and in general the broader view of analytical algorithms as operations, that lends positive interest to critical research concerning the convergence of series, the continuity and the derivability of functions, etc.; that is, to those speculations on the principles of the theory of arbitrary functions that were pushed forward by Riemann, Weierstrass, Dini, etc.

Seeing among the founders of this criticism those same mathematicians who constituted the body of the theory of analytical functions makes us understand the profound connection between two fields of study that are sometimes contrasted as two different branches in mathematics.

In reality, if analytic functions were born in Lagrange's mind to respond to doubts about the foundations of infinitesimal calculus, their progress always appears to be linked to critical concerns of the same kind; suffice it to recall that the most beautiful result of the theory of analytic functions is their determination by means of their singular points in the plane, and that the Riemann-Dirichlet principle lies at the basis of the existence theorems concerning them.

Now, in order to deepen our understanding of questions of existence in relation to the extended concept of functions, series, etc., a new analysis of the continuum is needed, that leads to an essential complement to the antique criticism. I am referring to the postulate of continuity and the new setup of the doctrine of irrational numbers considered in their entirety. This doctrine is in fact identified by Weierstrass with the general theory of the convergence of series, and by Cantor and Dedekind with the determination of the conditions of existence of limits. The intimate relationship between it and the concept of the arbitrariness of functions is revealed in Cantor's critical developments on sets and their cardinality.

Due to these developments and more recent speculations (by Veronese, Hilbert, etc.) on the so-called non-Archimedean continuum, modern thought seems to have taken the analysis that began almost 2,500 years ago with the Pythagorean hypothesis to its extreme consequences. Certainly, the concept of the continuum remains for us the same as that which Eudoxus and Archimedes placed at the basis of their constructions, but enriched with a new existential principle that is related to the extension of mathematical concepts; and the meaning of this principle is brought to light thanks to the aforementioned non-Archimedean developments, which have become an integral part of the critical exploration of the continuum.

§ 6. — The intensive development of mathematics: equations and imaginary numbers.

The above considerations aim above all at the *extensive* progress of mathematics, showing in relation to it the role of the critique of principles. Ideas initially suggested by a narrow intuition are refined through the analysis of conditions of validity and become capable of fertilising an ever-wider field of problems. This extension, which is an aspect of scientific progress, is presided over by critical thinking, understood as an instrument of positive knowledge.

However, the development of mathematics does not only take place in an extensive sense, but also in a direction that can be described as *intensive*.

Broadening the scope of problems by subjecting an ever wider field of real relationships to analysis does not exempt us from deepening our understanding of antique problems, continuing to seek effective solutions by specific means.

Thus, the more general consideration of irrational numbers gives way to a theory of rational or integer numbers or of particular types of irrational numbers; and while algebraic equations find their natural extension in differential equations, and these in partial differential equations, integral equations and integro-differential equations, each of these classes of problems gives rise to a specific intensive development.

The extended view provides this same development with a fundamental criterion, namely the principle of *relativity*, whereby the sought-after solution is related to given means and converted into a hierarchical classification of the various types of problems according to an order of increasing difficulty.

Now, mathematics, considered from this point of view, reaches its culmination in the development of algebra, understood broadly as a general theory of *qualitative* problems that arise in relation to the group of rational operations (algebraic equations and functions, elliptic and abelian functions, algebraic-differential equations, etc.), or as the first branch of a qualitative theory of functions.

The task of the criticism of principles that we have recognised in the extension of concepts and problems is no less essential with regard to this branch of mathematics, where, at first glance, it might appear less evident.

Let us return in our thoughts to the origins of the theory of algebraic equations. Second-degree equations are solved geometrically in Euclid's Book II, and their solution is connected to the discovery of incommensurables, which has already been mentioned above. Through the Arabs, this doctrine took on a properly algebraic form, which highlights the general problem of higher-degree equations.

From this representation negative numbers arose (first encountered by the Indian Bhāskara in 1114), later taken up by the mathematicians of the 15th and 16th centuries, Pacioli, Cardano, and Stiefel, and, after Harriot and Descartes, adopted as ordinal numbers or abscissas of a straight line.

The use of symbols was not yet familiar to Italian mathematicians of the sixteenth century, who, turning to the treatment of cubic equations, saw in them the underlying geometric problem. Scipione dal Ferro and Niccolò Tartaglia discovered the rules for solving these equations, which were then divided into three classes(4) ; and these rules were taken up and developed by Girolamo Cardano and Raffaele Bombelli.

The irreducible case of third-degree equations then paved the way for the consideration of imaginary numbers, i.e. the critical problem of the value and meaning that can be given to the square root of a negative number and its use in calculations. Further progress in algebra required that this delicate concept be fully understood; Bombelli's profound elaboration of it remained largely misunderstood until Leibniz and Wallis, and — taken up again by these mathematicians — it was fully developed with De Moivre's trigonometric interpretation; however, critics still struggled to find a concrete meaning for complex numbers, and succeeded with the well-known geometric representation of Wessel, Argand, and Gauss.

Only then, on the basis of the completed criticism, was the fundamental theorem that an equation of degree n has n roots established; this theorem had been sought by mathematicians of the eighteenth century, notably D'Alembert (1746), and was rigorously established by Gauss in 1789.

If we now reflect on the place that imaginary numbers occupy in function theory, we are led to understand the full extent of the value of the criticism that began with Bombelli's speculations and was prompted by a specific problem such as that of cubic equations.

(4) $x^3 + px = 9$, $x^3 = px + 9$, $x^3 + 9 = px$. See D. Gigli, *Dei numeri complessi...*, in F. Enriques, *Questioni riguardanti le Matematiche elementari*, Zanichelli, Bologna, 1912.

§ 7. — **Riemann's theory of algebraic functions and the criticism of the principles of geometry.**

Let us proceed to consider the history of the development of algebra, and we find that each step forward is equally linked to a criticism touching on the fundamental concepts of mathematical science.

While fourth-degree equations can be traced back to third-degree equations, the study of equations of a degree higher than the fourth leads, with Ruffini and Abel, to the demonstration of the impossibility of solving fifth-degree equations by radicals and thus to the general doctrine of algebraic solvability according to Galois. This doctrine, which was then extended in various ways, has fertilised all branches of mathematics, and with Lie it succeeded in providing the basis for a rational classification of differential equations. It is ultimately a critique of certain elementary concepts: order, operation or correspondence, group of operations.

The place of these concepts with regard to the principles of mathematics, and in particular geometry, is clearly evident in the work that, in many respects, can be recognised as central to the development of mathematics in the nineteenth century: the work of Bernhard Riemann.

The extraordinary creative activity of this thinker is illuminated by a brighter light for those who investigate the profound link between the research that gave rise to the general doctrine of algebraic functions and their integrals, his criticism of the concepts of calculus and that — of a more broadly philosophical nature — which touches on the principles of geometry and lays the foundations for it in *Analysis situs*. It is precisely the relationship between the invariant properties of algebraic functions under birational transformations and the connection(5) [*connessione*] of the corresponding Riemannian surfaces that constitutes the dominant discovery in that field of study(6).

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(5)NdT. We follow Enriques' terminology ("*la connessione delle corrispondenti superficie riemanniane*"), also adopted in the French translation of the article in the 1912 Supplement of *Scientia* ("la connexion des surfaces riemanniennes").

(6)NdT. This relationship reads precisely as follows, in Enriques' terminology. "L'ordine di connessione di una superficie di Riemann vale $2p$, designando p il genere della relativa curva algebrica, supposta dotata di singolarità ordinarie". Infatti "l'ordine di connessione della superficie di Riemann esprime un carattere della curva invariante rispetto ad una qualsiasi trasformazione birazionale" [Enriques, Federigo and Oscar Chisini, *Lezioni sulla teoria geometrica delle equazioni*

Furthermore, it is thanks to Riemann's synthesis that the pure speculations of non-Euclidean geometers are reattached to the organism of mathematical reality, forming the subject of history (quadratic differential forms). So, the relationship between the criticism of the principles of geometry and the development of mathematical doctrines will then appear more clearly in many respects. The crux of this relationship lies in projective geometry, which, from Poncelet to Möbius, Steiner and Staudt, developed not only as a doctrine of projections and a method of reduction, but also as a critique of spatial concepts and relationships, succeeding in a qualitative treatment of these independent of metric notions. The value of this development with regard to philosophical problems concerning principles is clear from the work of Beltrami, Schläfli, Cayley, Klein, etc. As for its importance in terms of the constructive progress of mathematics, suffice it to point to the new form given to problems concerning algebraic functions and the results that follow from them.

In fact, the followers of this Riemannian approach (notably Clebsch and Noether) have renewed the doctrine through a more abstract consideration of the projective geometry of algebraic curves and surfaces, which has finally led to a more general position on the same problems of algebra. Thus, for example, a system of equations satisfied by a finite number of solutions comes to be regarded as having an invariant degree for a continuous change of parameters, by virtue of the geometric convention that extends the space with improper points, and therefore leads to counting asymptotic solutions among the effective solutions and, in particular, to eliminating certain cases of incompatibility.

Now it is essential to note that not only projective geometry as a constructed edifice, but also the investigations into its foundations must be considered as an essential element of the new theory of algebraic functions. In fact, one of the main methodological concepts here is the abstract consideration of projective geometry as a hypothetical-deductive system characterised by postulates, i.e. the fruitful principle which, by generalising the duality discovered by Gergonne, allows us to consider certain systems of entities or functions as different interpretations of that geometry.

e delle funzioni algebriche. vol. I, Zanichelli, Bologna, 1915, p. 373]. English transl. "The order of connectivity of a Riemann surface is equal to $2p$, where p denotes the genus of the corresponding algebraic curve, assumed to have ordinary singularities. In fact, the order of connectivity of the Riemann surface expresses a characteristic of the curve that is invariant with respect to any birational transformation."

Take, for example, the search for finite groups of linear substitutions over a variable. From a geometric point of view, we are therefore dealing with groups of homographies on the straight line. But the totality ∞^3 of homographies on the straight line forms a linear system that can be considered in an abstract sense as an ordinary projective space; it is therefore sufficient to call the homographies themselves 'points' and 'straight lines' the 'pencils of homographies' [*fasci d'omografie*]. The representation that results, studied by Stephanos, highlights a quadric image of the degenerate homographies and a point that corresponds to the identity. Our problem is reduced to determining the groups of homographies that leave a quadric and a point not belonging to it unchanged. With an imaginary linear transformation, the quadric will change into a sphere and the point into its center. The groups sought will correspond to the groups of rotations of the sphere, that is (as is well known) to the groups of regular polyhedra!

Now the general principle of abstract projective geometry takes on its full extent thanks to the concept of multidimensional spaces, and thus it becomes possible to treat as "spaces", that is, to translate into the terms of general projective geometry, the series g_n^r of groups of points on a curve, the linear systems of curves on a surface, etc.

§ 8. — The new developments in modern algebra.

I have mentioned algebra in its geometric guise, which contemplates equations and systems of equations with multiple unknowns, in the face of birational transformations; and I do not believe I am indulging in personal preference when I say that it is now at the forefront of intensive progress in mathematics, as the legitimate continuation of the great tradition of algebraic problems, of which I have given a brief overview. All the more so since almost all branches of qualitative mathematics, from Abelian functions to automorphic functions and algebraic-differential equations (according to the developments of Poincaré and Painlevé), are intimately connected with it.

Let us therefore consider a theorem relevant to that theory, e.g. the theorem that the annulment of the genus provides the condition for the resolution of an equation

$$f(xy) = 0,$$

by means of rational functions of a parameter:

$$x = \phi(t),\ y = \psi(t),$$

(Clebsch), and that the actual resolution is obtained by rational operations and at most by extracting a square root from the coefficients of f (Noether). Such a statement provides a fully determined answer to an equally determined question; nothing could seem to be further from the field of criticism of principles; yet the history of that achievement presupposes, as we have mentioned, a long elaboration of concepts: from irrational to imaginary numbers, from permutations to multiply connected surfaces [*superficie più volte connesse*], from projective geometry to the generalised principle of duality that arises from the logical contemplation of the geometric edifice!

If anyone now considers that the critical elaboration of concepts is found in those algebraic doctrines only to establish a basis on which the construction will then continue without any further relation to criticism itself, let him reflect on other further developments; and thus be led to recognise how the invariant theory of surfaces requires a delicate analysis, tending towards the same purpose for which improper points are introduced into projective geometry, namely to remove exceptional cases of invariance (conventions on the holes of curves or on the base points of linear systems, on reducibility or irreducibility, etc.); and on the other hand, he will see that the same ideas dominating the extension of the field of numbers, i.e. the introduction of fractional, negative or imaginary numbers, find a fruitful application in the broader concept of functions ϕ of degree $n-4$, *adjoint* to an equation of degree n

$$f(xyz) = 0,$$

which are invariants of f with respect to birational transformations (Clebsch-Noether)(7). In fact, the extension of Clebsch-Noether's

(7) [NdT] In Picard's work, these results are accounted for as follows: "The theory of algebraic functions of two independent variables has been the subject of significant research, among which the memorable work of M. Noether (*Math. Annalen*, vols. II to XI) deserves special mention. The eminent geometer mainly studied the question from an algebraic point of view, deepening the study of these adjoint polynomials of order m-4 (m denoting the degree of the surface), which are analogous to the adjoint polynomials of order m-3, playing such an important role in the theory of algebraic plane curves. He thus arrived at the concept of two fundamental invariant numbers." [La théorie des fonctions algébriques de deux variables indépendantes a fait l'objet d'importants travaux, parmi lesquels il

invariant theory has been made precisely in the sense of considering, so to speak, the *virtual* ϕ which — in the case of the genus $p = 0$ — sometimes yield actually existing ϕ^n functions, which (at least for $n = 2, 3, 4, 6$) constitute *new invariants* of f. And let me remind you that thanks to these invariants, it has been possible to assign in a simple and determined form the condition for the rational resolution of $f(xyz) = 0$, the condition for the transformation of $f(xyz) = 0$ into an equation of two variables $F(xy) = 0$, the condition for $f(xyz) = 0$ to possess a continuous group of birational transformations in itself, etc.

Finally, the study of transcendents connected with a two-dimensional algebraic field confronted Picard and Poincaré(8)

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convient de citer tout particulièrement les mémorables recherches de M. Noether (Math. Annalen, t. II à XI). L'éminent géomètre a principalement étudié la question au point de vue algébrique, en approfondissant l'étude de ces polynômes adjoints d'ordre $m - 4$ (m désignant le degré de la surface), qui sont les analogues des polynômes adjoints d'ordre $m - 3$, jouant un rôle si important dans la théorie des courbes planes algébriques. Il est arrivé ainsi à la notion de deux nombres invariantifs (sic.) fondamentaux.] Émile Picard, « Mémoire sur la théorie des fonctions algébriques de deux variables », *Journal de mathématiques pures et appliquées* 4e série, tome 5 (1889), p. 135-319.

(8) [NdT] Enriques does not provide references, but it is likely that he refers, among others, to the 1889 work of Picard on the theory of functions of two variables quoted above. We follow and refer to Henry Paul de Saint Gervais (the name of a collective of mathematicians that created the *Analysis situs* website: `https://analysis-situs.math.cnrs.fr/`). "Picard demonstrates in his *Mémoire sur la théorie des fonctions algébriques de deux variables* that the first Betti number of smooth hypersurfaces of $\mathbb{P}^3(\mathbb{C})$ is zero. Unlike in dimension 1, only a fairly limited class of algebraic surface topologies can be obtained using this construction. For example, no tori appear in this way, nor do any complex hyperbolic varieties, etc. Some time later, in the third supplement [H. Poincaré, Sur certaines surfaces algébriques. Troisième complément à l'« Analysis sitûs », *Bulletin de la Société Mathématique de France*, Tome 30 (1902), pp. 49-70], Poincaré completes Picard's result and shows the analogous property for double ramified coverings of the complex projective plane. But he goes further, announcing that they are simply connected!" [Picard démontre dans son « Mémoire sur la théorie des fonctions algébriques de deux variables », que le premier nombre de Betti des hypersurfaces lisses de $\mathbb{P}^3(\mathbb{C})$ est nul. À l'opposé de la dimension 1, on n'obtient donc qu'une classe assez restreinte de topologie de surfaces algébriques par cette construction. Par exemple, aucun tore n'apparaît ainsi, ni aucune variété hyperbolique complexe, etc. Quelque temps plus tard, dans le IIIème complément, Poincaré complète le résultat de Picard et montre la propriété analogue pour les revêtements ramifiés double du plan projectif complexe. Mais il va plus loin, il annonce même qu'ils sont simplement connexes ! » Henry Paul de Saint Gervais, « Simple connexité des revêtements doubles ramifiés du plan projectif complexe », `https://analysis-situs.math.cnrs.fr/Simple-connexite-des-revetements-doubles-ramifies-du-plan-projectif-complexe.html`.

with the difficulties concerning the connection [*connessione*(9)] of four-dimensional varieties; and it can be assumed that certain difficulties that have not yet been overcome will be resolved on the day when the critique of the principles of geometry will have explored the highest problem that still appears unsolved in its field, giving a pure geometric basis to the edifice of *Analysis situs*.

§ 9. — Conclusions: pragmatism and mathematical naturalism.

The thesis stated at the beginning seems to me to have been sufficiently demonstrated: the critique of principles is an integral part of the history of the development of mathematics, both from an extensive and an intensive point of view; it is the process of elaborating and defining concepts that tends to extend the data of intuition to ever wider fields and thus to broaden the scope of problems and prepare more penetrating tools to provide definite answers to deeper questions.

Now, this historical view presupposes, in a way, a law of development of mathematics, in relation to which it assigns, so to speak, a natural finality to the critique of principles. Meanwhile, the progress of this critique itself seems, on the contrary, to give rise *to the unlimited arbitrariness of mathematical construction*.

We have already seen that functions, once assumed to be a given of natural reality (power, root, exponential, logarithm, sine, etc.), give way to general functions in the sense of Dirichlet, which are arbitrary correspondences. The fundamental properties of numbers no longer appear to be the expression of necessary axioms but, especially in the analysis of Cantor and Peano, become arbitrary conditions with which certain ordered sets are defined. For example, the principle of mathematical induction loses its value as a logical canon to represent only a constructive condition of the well-ordered series of objects to which the integers correspond, so much so that the negation of the principle gives rise — in that series — to the existence of limit points corresponding to transfinite ordinal numbers.

Geometry, which had seen its field of possibilities expand with the treatment of non-Euclidean hypotheses and multidimensional spaces, now becomes susceptible to unlimited extension, so that there is no longer any group of objects with any property that cannot claim the name 'space'.

(9) [NdT] Higher connectedness, in modern language.

The logical school did not fail to highlight the significance of the revolution that had taken place. Axioms have been dethroned; the spell of their divine right, that is, their foundation in an evidence or natural necessity of the human spirit, has been broken, and they have become mere postulates, no longer principles or members of a noble aristocracy, but elected officials of a democratic republic, who can be revoked or replaced for reasons of economy or simple renewal.

An Aristophanes might also find that unlimited freedom of choice risks converting this democracy into true demagogy; that *dishonest* functions too often take the place of simple but honest functions satisfying the theorems of infinitesimal calculus, that certain constructions of bizarre geometries (justified at first as a means of investigating certain relationships of subordination) affirm the freedom of the inspiring idea in the same way as the successive forms of government in the Principality of Monaco, under the auspices of Rabagas.

Yet even the somewhat baroque exaggerations to which today's criticism of principles leads have served to spread a correct idea of the value of logic and, conversely, allow us to guess the value that other non-logical elements assume in mathematical knowledge. The importance of this view of logic is already evident from the fact that this critical approach has given rise to a vast philosophical movement, which has spread in our own day under the name of *pragmatism*. In fact, the father of that philosophical pragmatism that has finally succeeded in an anti-scientific reaction is precisely the pragmatism of mathematical logicians who, armed with the critique of principles, claim that postulates are definitions and deduce the arbitrariness of mathematical construction, against a conception that could be called *naturalistic*, according to which mathematical entities exist outside of us, like living species in the natural sciences, as objects of discovery and observation.

Well, if logical-mathematical pragmatism succeeds in victoriously combating naturalism and the naive realism that underlies it, that pragmatism is in turn defeated by history. The history of the development of mathematics has shown us precisely the work of logical criticism in a centuries-old elaboration of concepts.

Therefore, to the consequences that one might draw from viewing postulates as implicit definitions, history responds that the definitions of mathematical entities themselves are not arbitrary, as they appear to be the result of a long process of acquisition and assiduous effort revealing some general motives of research.

There is a tradition of problems and there is an order that presides over the extensive and intensive advances of science; therefore, there is only one subject matter proper to mathematics, which definitions aim to reflect; hence, the arbitrariness of who defines does not seem different from that of the architect who arranges the stones of a building rising according to a harmonious design.

Mathematical science is in fact a work of architecture; it is not a reality that presents itself to the gaze of an external observer as something given, but a process that is created by the human spirit and yet reveals the very reality of the creative spirit.

Thus, the act of will that the mathematician claims to be ever freer in the positioning of problems, or in the definition of concepts or in the assumption of hypotheses, can never mean arbitrariness, but only the ability to approach from multiple sides, through successive approximations, some ideal implicit in human thought, that is, an order and harmony that reflects its inner laws.

If this is the conclusion that emerges from a historical view of science and criticism, logical-mathematical pragmatism, far from opening an era of fantastic constructions multiplying infinitely almost for fun or out of whimsy, will have given research a higher awareness of its purposes; and on the other hand, by purifying Logic, it will have demonstrated its insufficiency and the need to deepen the other psychological elements that give meaning and value to mathematical construction.

§ 10. — Mathematics as an instrument and model of science.

To our idealistic view, which seems to arise from an exclusive consideration of pure mathematics, others might oppose an apparently broader view in which mathematics itself is considered no longer as an object in its own right, but as an instrument of natural science. However, this concept (which may be suggested to some, as a reaction, by the exaggerations of logical pragmatism) would lead to a singular impoverishment of the field of mathematical activity. It would take us back to the view of Fourier, who reproached Abel and Jacobi for studying algebraic equations and functions instead of turning their attention to the movement of heat; a reproach to which Jacobi replied , that the sole purpose of science is the honour of the human spirit and that, in this respect, a question of numbers is no less valuable than a question relating to the order of the universe.

If it is necessary to provide arguments in support of Jacobi's view, it suffices, I believe, to reflect on how the difficulties of number theory have always attracted the highest intellects, including the founders of the system of the world and of mechanics.

But the narrowness of the above consideration of mathematics is best revealed on its own ground, thanks to an in-depth view of the place that mathematics occupies in the order of knowledge.

While the breadth of applications of calculus [*calcolo*[10]] may have reinforced the idea that mathematics is merely a *tool* of physical cognition, the greatest thinkers of all times have recognised it as a *model* of science.

This concept offers a more accurate view of the relationship between the progress of the mathematical spirit and scientific progress in general. Poincaré rightly points out that the mathematical spirit, independent of the power of algorithms, is active in Faraday's fruitful intuition of analogies, which led Maxwell to his memorable discoveries. In the same sense, it can be said, for example, that thermodynamics is entirely a mathematical work, although it is largely independent, not of concepts, but of the results of calculus [*calcolo*].

Mathematical inspiration is equally evident in other branches of science, which do not strictly speaking involve a phase of mathematical development.

In this way, it can be said that progress tends to realise the scientific ideal of Plato, Descartes and Leibniz, which places *mathematics as the model of science.*

This view, which broadens the value of mathematics in the universal order of knowledge, also restores the full value of the free development of pure theory, as claimed by the nineteenth century.

Now, in the light of the idealistic conception of mathematics, this view acquires a new meaning with respect to the *realist* position of the aforementioned philosophers.

When Plato constructed the world of Ideas in the image of the classification of geometric forms; when Galileo, Descartes

[10] [NdT] The term « calcolo » has a broader signification than mathematics education's "calculus". Our translation of *calcolo* should be understood accordingly: as a synthesis of "calculation", mathematics education's "calculus", and, more generally, "mathematical theories involving computations".

and Leibniz shaped a new dynamic reality, whose invariants are relationships of succession, i.e. natural laws; these philosophers projected the inner process of their spirit onto the external world, and in that world, they believed they recognised the elementary causes, as simple data of reality itself.

Today, epistemological criticism, connected to the investigation of the principles discussed above, warns us that the mathematical model of science has a different meaning; it is not a question of discovering the profound metaphysical structure of reality, but of recognising the forms of spiritual activity that shape sensible reality in scientific construction, according to the intimate laws of the human spirit.

Thus, mathematics, which for Plato, Descartes, and Leibniz offered the foundation of a philosophy of nature, rising to a grandiose rationalistic metaphysics, today, thanks to the powerful revival of contemporary criticism, gives rise to a new philosophy of the spirit, that is, an epistemology that must reveal thought to itself by investigating the profound psychological harmonies in which it is shaped in the continuity of history.

And from this point of view, the critique of principles promises to bring new and important results; after illuminating the true nature of logic, it will succeed in deepening the study of the intuitive elements of different kinds that give mathematics its inexhaustible value.

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