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History of mathematics and profound mathematical results: the post-Cavaillès debate in French epistemology

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§ — Introduction.

The present issue features English translations of two articles by Alain Michel: the first, titled "Mathématiques et 'profondeur': l'exemple de la théorie des nombres",(2) and the second, "Jean Cavaillès dans l'héritage de Léon Brunschvicg".(3) The two articles under consideration reveal the coexistence of two traditions in French mathematical philosophy, which differ fundamentally in their conception of history. The first tradition can be traced back to the work of Léon Brunschvicg (and, prior to this, Émile Boutroux) and posits that mathematical progress is the exploration of a vast, open array of possibilities. According to this theory, mathematical progress can only be conceived as a necessary development when viewed retrospectively, disregarding the intricacies of historical analysis. In the context of twentieth-century French philosophy, Alain Michel is a prominent figure representing this tradition of mathematical philosophy. The second tradition, as advocated by Cavaillès, seeks to identify the structure of a sort of creative necessity — yet within the context of the unpredictable evolution of mathematics. The concept of duality between operation and object was further introduced by Gilles Gaston Granger in order to develop this conception.

Our primary plan for this special issue involved the joint presentation of two articles to illustrate the two traditions under consideration. The first of these, entitled "What is a profound result in mathematics?",(4) was written by Gilles Gaston Granger. The second, entitled "Mathématiques et 'profondeur' : l'exemple de la théorie des nombres", was written by Alain Michel (see footnote 2). The aim of such an undertaking was to furnish a non-French-speaking audience with a concrete illustration of the implications of the debate on the nature of history of mathematics, by revealing the differences it gives rise to with regard to the epistemologically troubling notion of depth. The exorbitant price demanded by the publisher, Springer, for the republication of Granger's article

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(2) Alain Michel, "Mathématiques et 'profondeur': l'exemple de la théorie des nombres", *Jean-Toussaint Desanti, une pensée et son site*, Georges Ravis-Giordani (textes réunis par), Fontenay-aux-roses (Hauts-de-Seine): ÉNS Éditions, 2000, pp. 181-199.

(3) Alain Michel, "Jean Cavaillès dans l'héritage de Léon Brunschvicg". *Revue de Métaphysique et de Morale*, 1, (2020), pp 9-36.

(4) G.G. Granger, "What is a profound result in mathematics?" in E. Agazzi and G. Darvas (eds), *Philosophy of Mathematics today*, Kluwer (1997) pp 89-100.

rendered the project unfeasible. Notwithstanding this failure, we endeavour to offer our comparative analysis of the concept of depth based on the assertions made by Granger and Michel in their respective writings. By establishing a correlation between the theme of conceptual depth and that of the mathematician's creative work, these two authors illustrate brilliantly the two opposite poles of the debate on the nature history of mathematics having animated the French tradition. This debate is, it can be argued, not confined to the philosophy of mathematics in isolation, since it ultimately concerns the question of what knowledge is.

§ 1. — Circumstances and theoretical issues surrounding the notion of mathematical depth.

In order to establish a solid foundation for the subsequent discussion, it is necessary to briefly review the historical context and the intellectual underpinnings of the debate on the notion of depth. This discussion is deeply rooted in the relationships that were forged by two prominent scholars, Gilles Gaston Granger and Alain Michel, over a period of at least seven years. These relationships were cultivated within the context of the seminar on Comparative Epistemology led by Granger in Aix-en-Provence.

Gilles Gaston Granger (1920-2016), in common with Jules Vuillemin (1920-2001), Jean-Toussaint Desanti (1914-2002) and numerous others, was a student of Jean Cavaillès at the École Normale Supérieure on rue d'Ulm. Alain Michel (1946-2017), a student of Desanti at the École Normale Supérieure de St Cloud and a distinguished reader of Georges Canguilhem (1904-1991), was employed from 1989 on in the CNRS Centre d'épistémologie Comparative, which was founded by Granger at the Université de Provence (now Aix Marseille Université). The two articles on conceptual depth were presented in the same year (1997), although the publication of Granger's article ("What is a profound result in mathematics?") re-elaborates the text of an earlier lecture which, most probably, must have been presented to the audience of the Seminar of the Centre d'épistémologie Comparative, of which Michel was a member. "Mathématiques et 'profondeur' : l'exemple de la théorie des nombres" was presented, in honor and in the presence of Desanti, at a colloquium held in Bastia in June 1997, before being published in 2000. Michel therefore became acquainted with Granger's work on mathematical depth, either by reading the article directly or during the comparative epistemology seminar, and he appears to be responding here, albeit implicitly, to Granger's analysis by adopting the opposite view to a series of his theses. The critical significance of this work with regard to Granger's work is of particular importance because of its contribution to the wider assessment of the philosophical legacy of Cavaillès and the French epistemological tradition. It directly engages with the fundamental question of the position of the history of science in relation to philosophy; more specifically, the role of the history of mathematics for the epistemologist who aims to understand, through mathematics, the 'work' of reason.

Granger explicitly referenced Cavaillès's work in a minimum of three articles published prior to and following the publication of "What is a profound result in mathematics?". The publications under discussion are "Cavaillès ou la montée vers Spinoza",(5) "Jean Cavaillès et l'histoire",(6) and "Cavaillès et Lautman, deux précurseurs".(7) In the latter, Granger places significant emphasis on three aspects that link him to Cavaillès, understood in this sense as a precursor of his own attempts to develop a "rationalism based on a precise interpretation of the 'work' of science" (*ibidem* p. 301). The three aspects under discussion are as follows: Firstly, Cavaillès's thesis that mathematical objects are produced by a system of "gestures" governed by rules is developed by Granger in his work through the idea of mathematical work as a distinction and dynamic association of matter and form, i.e., of objects and structures. Furthermore, consideration must be given to the notions of paradigm and thematisation developed by Cavaillès. These notions constitute the moments of the double movement that renders structures explicit. Granger's concept of duality reveals the presence of the aforementioned elements, albeit in a transformed form. Finally, it is important to consider Cavaillès's conception of the history of *mathematical experience*, understood as a system of actions where concepts are the results brought into existence by mathematical work. In such a history of the *mathematical experience*, it is possible to distinguish a subjective aspect, linked to the "consciousness" of the mathematician who produces them, as operations of isolated thoughts, and the true mathematical concepts, results of "a

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(5) G.G. Granger, « Cavaillès ou la montée vers Spinoza », *Les études philosophiques*, 2, (1947) pp271-279.

(6) G.G. Granger, « Jean Cavaillès et l'histoire », *Revue d'Histoire des sciences*, 49(4) (1996), pp569-582, réédité dans *Philosophia Scientia* 3 (1) (1998), pp569-582.

(7) G.G. Granger « Cavaillès et Lautman, deux précurseurs », *Revue Philosophique,* 3(2), (2002), pp293-301.

systematization of operator acts, introducing contents and [going] beyond acts proceeding from a subject" (*ibidem* p. 300) This interpretation of Cavaillès will be developed in full by Granger though the concept of style (see below).

It is also known that Granger rarely references the work of Léon Brunschvicg (1869-1944), one of the professors of Cavaillès and Lautman, despite the significant influence he exerted on his students' work. Additionally, Granger's analysis overlooks the contributions of Canguilhem, also Brunschvicg's fellow student and Cavaillès's contemporary.(8)

Michel, whose dissertation, *Étude sur la constitution et les développements de la théorie moderne de l'intégration* (*Study on the constitution and development of the modern theory of integration*),(9) directed by Desanti, was explicit in one of his last articles, written in 2016, "Jean Cavaillès dans l'héritage de Léon Brunschvicg" (*Jean Cavaillès in the legacy of Léon Brunschvicg*),(10) about the assessment he felt should be made of Cavaillès's work. Brunschvicg and Cavaillès agree, according to Michel's analysis, on the thesis that history is the only means available to the philosopher to understand the mind (Brunschvicg) and to construct a theory of reason (Cavaillès), yet what separates Cavaillès from Brunschvicg essentially concerns the question of the status of the modality of historical judgments: necessity according to Cavaillès, possibility according to Brunschvicg. The question is crucial. For Cavaillès, if the historical analysis of mathematics enables us to identify stable meta-categories in accordance with which the progress of reason can be outlined, then it is possible to speak of a certain necessity in the progress of mathematics itself. This necessity, for which Cavaillès claimed to have found an essential exemplification in Dedekind's work, is not entirely opposed either to the idea of a contingency in the conditions of production, proper to the historical context in which mathematical concepts emerge, or to the idea of an unpredictability in the results, due to the inevitable opacity of mathematical objects. The crucial point is that the emergence of

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(8) In private conversations, Michel spoke of Granger's distrust of both Canguilhem's and Desanti's thoughts.

(9) A revised version of this thesis has been published as *Constitution de la théorie moderne de l'intégration*, Vrin (2002).

(10) A. Michel, "Jean Cavaillès dans l'héritage de Léon Brunschvicg". *Revue de Métaphysique et de Morale*, 1, (2020), pp 9-36. (English translation in this issue).

new concepts is always the result of the internal dynamism inherent in mathematics: it's a necessary generation of new objects from the position of new operations made necessary by the emergence of new problems. Such a process is necessary in the sense that it is not plural (the negation of a mathematical thesis to give rise to an alternative development is impossible).(11)

On the other hand, for Brunschvicg, as he would be for Canguilhem in his own chosen field — philosophy of life sciences — scientific progress is achieved by exploring an open field of possibilities where only a retrospective gaze, inattentive to the details of historical analysis, can claim to see necessity. In short, the history of mathematics, no less than the history of anything else, is a non-necessary process, hence its irreplaceable centrality as a terrain for philosophical analysis.

This comparative analysis of the relationship between Brunschvicg and Cavaillès, which Michel develops in his 2016 article as part of his comprehensive and profound assessment of Cavaillès's legacy, represents the explicit moment when a series of remarks that punctuate Michel's entire work converge. For this reflection, "Mathematics and 'Depth'" is a significant step. In this sense, Michel's relationship with Granger may well have been the catalyst for a profound reflection on the issue that divides the two traditions in French epistemology that grew out of Brunschvicg: Cavaillès-Granger tradition and Canguilhem-Michel tradition.(12)

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(11) For a completely opposite position, see J. Vuillemin, *Philosophie de l'algèbre* PUF (1993) § 56, p 505 (first edition 1962).

(12) Note that in the article "Jean Cavaillès dans l'héritage de Léon Brunschvicg" (*op.cit.*), Michel takes up in its entirety the final part of his article of 1997 on Mathematics and conceptual depth about Kronecker's work (opposed in value to Dedekind's work on the theory of ideals). Moreover, the questions raised by this analysis are taken up again in 1998, in the article "Après Jean Cavaillès, l'histoire des mathématiques" (*Philosophia Scientiae*, 3 (1), 1998, pp 113-137), where Kronecker's "arithmetical intuitionism" is again referred as a reason for reconsidering in depth some of Cavaillès's conclusions about the nature of history. What's more, Michel's 1995 tribute to Granger, in which the historical material analyzed to test the concepts of duality and invariance central to Granger's work comes from his *thèse d'état* on the theory of measurement, in turn ends with a sentence that once again summons Cavaillès and Granger to the question of history: "If the philosopher can attempt to describe, in categories, *a priori*, 'not anterior to science' but 'the very soul of science' according to the terms Cavaillès used in connection with Bolzano, he must at the same time agree that these remain subject to the conditions engendered by their operative investment, which amounts to recognizing that they are constituted in a history, implying for their content unforeseeable modifications". "Epistemological analysis and the history of mathematics" in *La*

It appears that Michel's implicit criticism of Granger is directed towards Granger's double departure from Brunschvicg. The first of these departures occurred with Cavaillès, and was based on the thesis of mathematical necessity. The second departure occurred from Cavaillès, and was due to the idea of a general theory of style that would encroach on historical analysis.

§ 2. — Mathematical depth: two different analytical frameworks.

Our two authors agree that the term 'depth', used along with other terms such as beauty, elegance and power, often appears in mathematicians' comments on the work of their own colleagues. This term (and the corelated adjectif 'profound') is generally used to indicate a result that occupies a key place in the research of their discipline. It is therefore an indication of a "spontaneous epistemological analysis", as Alain Michel calls it in his article, a reflexive gesture by a community on its past and present practice, which for this reason deserves to be analyzed. Indeed, it reveals an appreciation of the value of mathematical conjectures and results as moments in the mathematical creation, — a conceptual creation, but a creation nonetheless. This value should be the subject of a consensus, given that it relates to the intrinsic and objective properties of mathematical work.

The question now is: can the use of this term in mathematicians' commentaries on past mathematical works give us important information for the internal knowledge of mathematics, which is crucial for the philosopher interested in elaborating a theory of reason? And more generally, is it possible to grasp conceptually (beyond the usage in this or that text given by this or that mathematician) the characteristics of mathematical depth? After the question has been formulated, it is worthwhile to draw attention to the divergent frameworks within which our two authors conduct their investigations.

For Granger, the problem of mathematical depth belongs stylistics. The question is in fact "related to a mathematical result or even to the position of a problem insofar as it is a *work*, that is to

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connaissance philosophique. Essays on the work of Gilles-Gaston Granger. J. Proust and E. Schwartz (eds), PUF (1995) p 210.

say the way in which a form and a content are found to be created and placed in relation" (p. 89). We have already mentioned the importance of this activity of distinguishing and relating forms and contents, which leads to the creation of structures and objects. Here, however, we propose a very explicit passage from the conclusion to the second edition of Granger's *Essai d'une philosophie du style*, (*Essai for a philosophy of style*), which will help us to understand the relationship between epistemological inquiry, historical inquiry and stylistic inquiry according to Granger.

"In the field of mathematics, one soon realizes that the philosopher's curiosity is directed in three directions. A philosophy of mathematics is first and foremost a commentary on the history of concepts. Whether viewed from the perspective of the technological evolution of mathematical tools, or from the evolution of the categories constitutive of the object, this commentary remains closely tied to history. But a philosophy of mathematics is also, secondly, a commentary on the internal systematicity of theories. Epistemology is then linked with logic. Finally, there remains the direction of stylistics. Here, it is the creation and use of structures that come to the fore. Not, to be sure, the historical development that leads to inventions, but the relationship between form and matter — latent or explicit, clear or obscure — that conditions, accompanies, or limits the establishment of a structure and its effective use. Such an analysis applies to all the major moments of mathematical thought; it seems particularly indispensable in the many cases where the same structure appears in different modes, introduced and used according to different styles. Such is the case, for example, with Newton's calculus with fluxions and Leibnizian infinitesimal analysis. But in all cases, it is the stylistic point of view that can give a philosophy of mathematics its concrete rational dimension — that of an essential *work* — which the analysis of systems obviously neglects, and which historical analysis can offer only a cursory [*cavalière*] and often misleading view of"(13).

We confine ourselves to highlight three aspects of this extensive passage, which appear to explicitly elucidate the role that stylistics is expected to fulfil in relation to the other facets of philosophical inquiry into science. Firstly, according to Granger, historical analysis is confined to an examination of the evolution of symbolism (mathematical tools) and the constituent categories of objects (the

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(13) G.G. Granger, *Essai d'une philosophie du style.* Odile Jacob (1988) p. 298. First edition Armand Colin 1968.

operational, structural tools that symbolism must express with the greatest clarity). These two aspects are also stressed in the analysis of three historical examples given in the article on depth. In this respect, Granger is faithful to Cavaillès's on the fact that the symbolic shaping of "gestures" (the structural operations within which matematical objects are defined) is essential for the philosophical inquiry. Secondly, logical analysis is essential, as it enables us to analyse the power, and therefore the fruitfulness, of the theories established over the course of history. Relationships of conceptual dependence, both internal and external to a given theory, are fundamental elements in the philosophical analysis of the evolution of mathematical thought. Finally, the stylistic dimension is crucial, says Granger, because the contingency of the contexts of discovery and the opacity of the objects themselves make the *same structure* appear in different ways. Here we find one of the consequences of the idea of a sort of necessity in the becoming of mathematics. In Granger's conception, the possible different mathematical structures emerging at a given moment for solving the same problem are and must be *the same*, modulo differences in style. In fact, only stylistic differences introduce plurality, without touching, so to speak, the singular essence of the mathematical objects invented.(14) Mathematical progress in the generality, clarity and deductive power of "invented" theories can only lead to unique

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(14)Invention, as Granger emphasizes in his 1997 article, must be understood in the double sense of discovery and construction (p.99). A metaphor from the later book *Science et réalité* may help us to understand Granger on this point: "A house is the *creation of* the architect and the mason; but once built it is real, independently of the activity of its creators; it could only exist, however, because its structure and the nature of its materials conform to the conditions deriving from the laws of physics and chemistry, necessary but not sufficient conditions for its reality. Applying the metaphor to mathematical objects, we would say that they are indeed produced by the mathematician's conceptual imagination, moderated by the logical requirements necessarily satisfied; but once produced under these conditions, these objects, which *have become real*, possess properties that are difficult to deduce, some of them even unpredictable, or non-deductible" (*Science et réalité, Odile Jacob* 2001, p.116). Kurt Gödel, in his 1951 lecture (*Kurt Gödel, Collected Works vol. III*, 1995, pp. 304-323), had already discussed one aspect of this kind of argument in relation to the consequences of his incompleteness theorems, emphasizing the strangeness of a situation in which the creator of an object finds it theoretically impossible to know certain properties of the object created, being able to recognize them only by defining more complex and abstract sets of rules (G. Crocco, "Gödel and Carnap again", in Paola Cantù, Georg Schiemer, *Logic, Epistemology, and Scientific Theories — From Peano to the Vienna Circle, Springer* 2023).

structures of objects of reason. This is Cavaillès's demand, which Granger makes his own, while at the same time, thanks to stylistics, he abolishes from the outset any discrepancy between what epistemology prescribes and what history reveals. In a sense, any possibility of being put to the test by history is neutralized by the idea that history can only be "cursory" and "misleading" in its analyses when we are looking for the "concrete rational dimension" of mathematics, i.e. the uniqueness of structures in the plurality of their creative styles.

Turning to Michel's chosen framework for analysing the notion of depth, it reveals quite a different approach. The aim is to use historical analysis to test the theses put forward by the philosopher-epistemologist to explain 'how mathematics is produced', and to use this explanation to elucidate the notion of mathematical depth. The question is whether, when applied to history, epistemological analysis, however precise it may be, leaves an element of ineliminable "residue" (says Alain Michel in his article), or whether the philosopher-epistemologist has already said everything possible with his own categories. Which philosophical-epistemological framework will Michel use for answering such a question? Desanti's one, for two reasons. Firstly, Desanti's philosophy of mathematics has spontaneously made room for the concept of depth in his attempt to reconstruct the 'movement of knowledge production' (p. 184). Secondly, because his concept of 'reactivation' incorporates history (with its 'unprecedented paths of passage') into his analysis, thus allowing effective historical analysis to proceed.

This is certainly not the place to summarize Desanti's analysis. In just a few pages, Michel says what is essential for his own purposes. To help with the reading, we might simply point out that Desanti is essentially interested in analyzing "mediations", i.e. the processes by which an elementary mathematical theory is generalized and consequently recast in more abstract terms. The notion of reactivation, in the analysis of which Desanti gives way to the notion of depth, is linked to the reconfiguration of links between successive layers of operative spaces of possibility that these generalizations liberate, and to the way in which certain "profound" generalizations enable the construction of unprecedented ways of passage between hitherto dissociated domains. Using the concepts of Desanti's philosophical framework and in particular the idea that the process of abstractive generalization (*mediation*) is the driving force behind mathematical creation, Michel points out the existence of a bifurcation, a residue that epistemological analysis seems to overlook, and which calls into question the absolute positive value judgement concerning Dedekind's work on ideals, as expressed by Cavaillès and Granger, and their underestimation of Kronecker's programme.

§ 3. — Three figures of mathematical depth and two opposing conclusions.

Beyond these differences in framework, the way in which the two authors organize the analysis of examples intended to show the general characteristics of the notion of depth is quite similar.

Granger's analysis. Granger identifies three archetypal forms of depth, which are formed by combining two themes that are frequently mentioned by mathematicians. The first is the theme of unpredictability, i.e. the unexpected nature of certain mathematical results, which impose their "sui generis reality" on us as a surprise. The second is the theme of foundation, which can be reached or guessed, but is capable of driving the reorganisation and clarification of an entire field of mathematical activity in both cases. The three characteristic figures are thus obtained by the greater or lesser emphasis that one or other theme can have in the resulting combination.

In the first figure, the characteristics of opacity and unpredictability take precedence in the analysis of the ζ function from Euler to Riemann.

Firstly, Granger draws on comments by Euler himself and André Weil to highlight the recurrence of terms such as "hidden", "marvellous" and "profound" which occur in the historical progression of the analysis: from the initial computational problem involving series, to the reflection on the function ζ which Euler defines on what today we would call real numbers, to the connection that Euler manages to establish between ζ and the prime numbers. Euler was fascinated by the mysterious and marvellous nature of prime numbers, and he was determined to understand their properties, despite the computational dissymmetries that obscured them. In particular, he was interested in the problem of the convergence of

$\zeta(s)$ for s an odd integer, which is far more difficult than the problem concerning even integers.

Then, Granger considers the historical development from the analysis of Riemann's conjecture about his function ζ (applied to the complex plane) to Dedekind's work exploring its application to algebraic numbers, opening the way to modern class field theory,[15] Granger highlights the astonishing distance between Euler's results and those of Dedekind, who forms the basis of a highly fertile development between algebra and number theory. This distance illustrates "the theme of a progressive, laborious and unexpected revelation of mathematical objects with other objects of an apparently greatly distant nature", which echoes the wonder felt by Euler. Granger also underlines how conceptual depth, which is corroborated by the fruitfulness of developments based on Dedekind's results, is always to be found in conjectures or results, and not in the calculations that may support them. Moreover, these calculations may be qualified as difficult and involve a high degree of skill, but they are never said to be profound.

The second figure of depth, in which the foundational character is emphasized, is illustrated by two examples.

The first example (which makes extensive use of Bourbakian analysis) concerns the development of a result by physicist **George Gabriel Stokes** (1819-1903). The modern formulation of the theorem is given by an elegant formula :

$$\int_{\partial c} \omega = \int_{c} d\omega$$

which should read as follows: "the integral of a *differential form* ω of whatever degree upon the boundary of a *differential manifold c* is equal to the integral of its *external derivative* on the *boundary of the manifold*". Granger highlights three aspects of this theorem in his analysis. Firstly, the transition from Stokes' formulation to the modern one highlights the "standardization of meaning" that this result brings to the concepts of differential form and external derivative in relation to gradient, rotational and divergence operators. Secondly, this standardization of meaning relies on the introduction of certain "radical" concepts, capable of expressing more deeply the general operative content of these three operators, and accounting for algebraic, analytic and geometric aspects alike. These new concepts are

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[15] Here, Granger essentially draws on the work of Blanchard and Rowe and McLeary.

those of variety, boundary and differential form, which Granger sees as constituting a more fundamental universe of objects. Finally, according to Granger, the deeper meaning of the theorem emerges from the conceptual duality highlighted by this result and made explicit by the formula that represents it. This duality concerns the concepts of differential form and external derivative. Granger uses this first example to emphasise that profound results, which lead to the introduction of new concepts, do not come from generalisation, but rather from elucidating structures that had remained 'unnoticed' until then. This elucidation process make it possible to 'ground' previously scattered facts by explaining their differences and similarities.

The second example of this second figure of depth is the Stone-Weierstrass theorem. The first version of this theorem was established by Weierstrass in 1885 and concerns "the polynomial approximation of a continuous real function of a real variable on a closed limited interval", and he adds "It is a question, therefore, of a specific sentence which authorizes in a certain way a *calculation* of approximation".(16) On the contrary, the form Stone gives it in 1948 shows "the 'profound' nature of the relationship between continuous functions and the set, structured as an algebra, of the polynomes, the terms of which are no longer simply the powers of the variable, but are themselves the powers of real continuous functions suitably chosen" (*ibid.*). Concerning subalgebras of functions, the theorem thus goes beyond the original question of approximation to elucidate the structure of the system of bounded continuous real functions. It is therefore a theorem which, rather than generalizing, simplifies an entire field of investigation.

In Granger's third and final figure of depth, opacity and foundation are mobilized in equal measure to illustrate how a fact, which at first appears surprising and strange, can give rise to research that enables the creation of new objects and thus profoundly renews the ground on which the previous theory rested. We start with the well-known result of the non-uniqueness of the decomposition of certain rings of integers into prime factors, and end with Kummer's definition of ideals and Dedekind's results, with the unexpected exploration of the universe of fields and modules. Granger ends the analysis of the third figure of depth as follows. :

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(16) *op. cit.* p.97.

> “Thus the fact of non-unicity discovered by Kummer is profound. It leads to the construction of these new objects that are the ideals; moreover, this “invention” — in the double sense of discovery and construction — is the origin of a recasting of basic concepts which will constitute contemporary algebra. One sees how, in this third figure of depth, the element of surprise before an unexpected mathematical fact is associated with the element of creation and exploration of a virtual universe extraordinarily fruitful of new facts”.(17)

Granger’s conclusion is entirely internal to the categories of virtuality and reality, which he had also established some years earlier in *Le probable, le possible et le virtuel* (1995), through a long meditation that drew on his work on Aristotle and Wittgenstein. The notion of depth would reveal the thickness of mathematical *virtualities* that are actualized in *realities*, once a form is given to them by mathematical work, once the mathematician’s genius has ″created and discovered″ them. It seems to us that this conclusion can only be understood in the context of Granger’s assertion of the *sui generis* necessity of mathematical objects. As Granger explains in his article ″Determinism and Necessity″(18), there is a close link between the systemic (structural) nature of the operating systems in which mathematical objects emerge and their ″requirement of existence″. We’ll end this section with a particularly eloquent passage from the latter article, before turning to Alain Michel to conclude our presentation.

> ″The specificity of mathematical necessity, and hence the logical character [of its proofs], comes from the profoundly *systemic* reality of objects. By this we mean that the mathematical object is defined only by its *insertion* into a system of presuppositions, or by its unexpected *emergence* within a system. A system that is undoubtedly coherent, or at least whose coherence, although originally imperfect, will soon become effective. The irrational number, for example, is initially an unreal object, but one whose existence is required by the arithmetic operating *system*, and which will later be integrated as a legitimate member

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(17) *Op. cit.* p. 99

(18) G.G. Granger, ″Déterminisme et nécessité″, in *Philosophes en Liberté*, Ellipses (2001) pp 9-23,

> of the new coherent system of real numbers. This coherence, the logical aspect of the necessity of existence, is often only achieved after the fact. The effective and fruitful aspect of this necessity of existence does not derive from it, but from the *systematicity* in which the object has an operative meaning. When infinitesimal calculus was born, for example in its Leibnizian version, infinitesimal quantities were *necessarily* introduced to give meaning to integration and differentiation operations. The new calculus forms a system, operatively though not yet logically coherent, and it is from this inclusion in a system that comes the necessity of the links between these new objects, and their own existence as necessary virtualities" (*op. cit* p15).

So, for Granger, rather than revealing a philosophical grammar (in Wittgenstein's sense), the precise analysis of mathematical texts through the notion of depth sheds light on a form of "moderate ontology" adapted to mathematical realities whose virtuality does not exclude conceptual thickness.

Michel's analysis. Michel appears to adopt Granger's three archetypal figures quite closely, as his historical analysis revolves around examples designed to demonstrate the same three "characteristic thematic components of depth". The first of these components is unpredictability, which is reduced, as in Granger's case, to the opacity of mathematical objects. The second component is generality or abstraction, which, as in Granger's case, is considered to enable the reunification of previously disconnected mathematical domains. The third component is the fruitfulness of a new foundation, which is introduced on the basis of remarkably influential works.(19) Michel analyses them through three examples drawn from Granger's treatment of the third example on number theory. He follows this exemple right through to the end of his article, aiming to demonstrate the potential of historical analysis in testing epistemological theory.

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(19) Nevertheless, it is worth noting that Michel often uses the conditional mode, probably to mark a critical distance from these three components. He asserts that they can only be a crude grouping (p. 184) of themes typical of the notion of depth. Perhaps the quotation marks in the title of his article are also an expression of this distance.

The first example on the theme of unpredictability concerns Fermat's method of infinite descent, which Fermat himself described as a "singular path" that enabled him not only to demonstrate difficult propositions, but also to formulate conjectures, including his famous last great conjecture, demonstrated by Andrew Wiles in 1994. In any case," adds Michel, "the method of infinite descent is an authentic mathematical creation, and nothing gave Fermat the slightest inkling, at the time of its creation, of his astonishing fruitfulness throughout history".

The second example, concerning abstractive generalisation, concerns Kummer's ideal numbers, which are considered a 'paradigmatic example of mathematical depth'. Alain Michel traces Kummer's development from his early work on cyclotomic integers to his ideal prime divisors, which were introduced to re-establish the uniqueness of decomposition into prime factors — demonstrated for integers by Fermat's method of infinite descent — for certain number systems such as the so-called non-Euclidean rings of integers. It is precisely to re-establish this uniqueness that Kummer extends his creation of ideal numbers to introduce ideal divisor numbers. The third example is dealt with very briefly and concerns Dedekind's set-theorethic development of Kummer's theory, which was mentioned above in the analysis of Granger's third example. In the final section, Michel suggests his disagreement with the highly positive evaluations of Dedekind's work shared by Cavaillès and Granger, as well as their underestimation of Kronecker's project.

Rather than presenting Michel's historical analysis, which is covered in more detail in his 1998 article *"Après Cavaillès, l'histoire des mathématiques"* (op. cit.), we would like to offer two concluding remarks on his analysis.

Our first comment relates directly to history.

We conjectured above that there were two implicit points of disagreement between Michel and Granger. The first concerned the question of mathematical necessity, a thesis Granger shared with Cavaillès; the second concerned the idea of a general theory of style encroaching on historical analysis.

Michel's lengthy comparative historical analysis of Kronecker's and Dedekind's works seems exactly intended to show that both authors share the same concern for generalization in relation to Kummer's work, but do not arrive at the same structure.

Dedekind's generalisation takes the form of a set-theoretic analysis, as opposed to the calculative tradition of Gauss and Kummer. His approach is not dictated by an internal necessity within mathematics. Dedekind's approach is to prioritise non-constructive definitions involving actual infinite sets. In contrast, Kronecker's method is to define objects implicitly through effective and algorithmic criteria of equivalence or congruence. The difference in these approaches reveals philosophical preferences rather than stylistic choices. These preferences must be evaluated not so much in relation to what these mathematicians say, but in relation to what they do mathematically. Now, Dedekind's reception of Kummer's work, which Michel illustrates in detail, and Hilbert's reformulation of the works of Kummer and Kronecker, have, in a sense, eliminated the constructivist choice and the algebraist point of view that characterised them from the mathematical scene, rendering them obsolete. It would be a long time before this hasty judgement was reversed.(20) This has happened because the formalism of mathematics and the way it is taught can direct our gaze and obscure other conceptualisation possibilities. Only a careful historical study can lead us to reassess the importance of Kronecker's work and remind us that another path can be taken from Kummer's work, rather than leading us to misleading and cavalier conclusions. Only history, with its meticulous analysis of mathematical works and the progression of results from one text to another, can help us to understand the mathematical value, not to be reduced to the ideological dimension, of other possible paths. In short, the history of mathematics cannot be likened to a long river, as the positivist view would have it, in which results accumulate linearly until they reach the sea of our knowledge via a large delta. Instead, it is more like an ever-changing watercourse that forms an estuary where seemingly dead branches can suddenly be reactivated by the main current, while those long considered the richest and most powerful can dry up.(21) Rebuilding these complex networks, without immediately assuming (as Granger would like to do with his stylistics) that the structures that emerge are and will remain unique, means accepting that human creativity cannot be

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(20) Alain Michel cites the works of Erich Hecke, Herman Weyl, Karl Ludwig Siegel and André Weil for the profound studies of mathematicians who were able to read and analyze Kronecker's works against the *doxa*.

(21) "In mathematics, as in all scientific matters, [the] lesson of history must be, in the words of George Canguilhem, conjugated in the reflexive past tense" A. Michel "Après Jean Cavaillès, l'histoire des mathématiques" (*op. cit*) p. 134.

constrained by the straitjacket that the philosopher-epistemologist would like to impose on it. ″Necessitarianism″ and stylistics are, in our view, the two aspects of Granger's philosophy that are called into question by Michel's historical analysis.

The second point concerns Alain Michel's constructivist attitude, which runs through the entire article. His insistence on the centrality of computations, algorithmic thinking and actual constructions in the creative movement of mathematics plays an important role in his critique of the ″necessitarian″ position. In conclusion, we feel it would be important to assess its scope, even from the point of view of those who do not share this demand. To do so, let's briefly recall how it plays a part in aspects of the analysis of the notion of mathematical depth.

Firstly, in commenting on the notion of unpredictability, Michel explicitly quotes Granger and his concept of duality(22) and recalls that according to Granger, unpredictability can be explained by the opacity of mathematical objects. Nevertheles he points out that this phenomenon appears very early in the history of mathematics, as soon as the rules and the objects have been laid down, i.e., as soon as there is calculation, even before Greek mathematics. Quoting Caveing, Michel suggests that, as soon as there is calculation (i.e., standardized operative rules performed on the objects of a real operative domaine) we have two types of properties:

> "those of the objects in the field, which can prohibit certain acts, such as subtraction or inversion, and those of the operations themselves, which above all determine the sequence of acts. Each object manifests itself as an operative singularity, which, authorizing such and such an operation with such and such another, or prohibiting it, is said to possess such and such a property" (p187).

But clearly, this alleged objectivity of objects' properties, which would belong to them as such, is nothing but a delusion, according to Michel:

> "A property that is in fact that of the domaine is converted (by the subjectivity of the mathematician-calculator) into that of the object. It is the properties of the domaine that regulate the operations that take place in it, and

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(22) The analyses of Desanti and Caveing are also called upon here to explain this phenomenon of opacity.

> therefore also the way in which objects come into being" (*ibid*).

At first glance, one might think that we are entirely within Granger's framework of analysis, but this strong reference to calculation is a sign that Alain Michel's constructivist and "algorithmic" conception of mathematics is leading him to severely criticise the positions of those who neglect or scorn the inductive importance of computations in the creation of profound mathematical results. The crucial point, which is independent of any algorithmic requirement, seems to us to be the following: unless one attributes a metaphysical reality to the notion of an operative domain, or uses the notion of virtuality in a sense that goes far beyond the virtuality of rule-based computation, one must always keep in mind that we are dealing with concrete operative domains — with acts of construction of conceptual thought in its computational operativity. Without the computational, algorithmic aspect, the analysis of systems of operations seems to require either the analysis of the subject who performs them, or the analysis of the objects these operations are aimed at.

Finally, in his analysis of Kummer and Kronecker, on the question of generalization and fecundity, Michel insists that during the generalization process: "There is a foundation, if you like, but this foundation is operative, not logical (set-theorethical)" (p187). If there is idealization, he adds, its shaping remains constantly operative, and indeed the context of Kummer's invention is constantly a context of use. For this reason, the analysis is inductive and essentially exploits the results of calculation, since abstraction is built by Kummer from algorithms, and in particular from that of divisibility. Depth here is essentially an operative gesture, based on the astonishing results of a calculation. Starting with a computational anomaly, the creative process begins to find its solution in new operating rules. The fruitfulness of a theory should therefore be judged by the extension of its own operational capacities, and not by its deductive force alone. The crucial point of Alain Michel's analysis here seems close to a consideration that Kurt Gödel makes in a completely different field. In the drafts (written between 1951 and 1959) of an article he devoted to Carnap's thesis that mathematics is merely the syntax of language, Gödel remarks that calculations are to mathematics what experimental protocols are to the natural sciences. In this sense, mathematical theories can be refuted in the same way as natural sciences. Even from a realist point of view such as Gödel's, it's fair to emphasize the importance of algorithmic thinking in mathematics, because without computation we can't ″test″ or ″disprove″ mathematical theories in their observable consequences. Hence the enormous progress, highlighted by Michel, that the use of computers has enabled in this field. However, it should be noted that it is necessary to understand what the machine is doing and to contextualise anomalies or refutations of conjectures within an appropriate theoretical framework. To achieve this, we must move beyond step-by-step algorithmic analysis to gain a global view of computation — a feat that the machine seems incapable of achieving. This raises the problem of carefully assessing what computer-assisted proofs can enable us to accomplish, which Michel could not have known about in 1997. In any case, the end of the article offers an intriguing and unexpected insight into the epistemological issues raised by the technological evolution of computer science, worthy of reflection.

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