Annals of Mathematics and Philosophy
***vol.* 1, nº 1**, 2023

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What is Axiomatics?

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Abstract. The article investigates axiomatics as a complex mathematical practice whose inquiry, while taking its cue from the analysis of some specific mathematical theories, requires an interdisciplinary approach. Axiomatics, if analyzed in detail through a study of its foundational component, of the styles with which it is associated and of the rules that govern it, performs a plurality of functions. It serves heuristic, descriptive, genetic-historical, pedagogical and architectural aims. But it can also play the role of conceptual analysis, modular analysis and coordination tool, soliciting a quest for rigor. An example taken from Peano's investigation of axiomatic systems illustrates the kind of results that this interdisciplinary approach to mathematical practice might produce, showing what can be achieved by considering axiomatics as research on the foundations of mathematics, or as a mathematical style, or as a social institution.

§ 1. — Introduction.

What the relation between mathematics and philosophy should be is a delicate question. Stewart Shapiro has distinguished two different modes of relationship between philosophy and mathematics, described respectively by the methodological principles *philosophy first* and *philosophy last-if-at-all*: in the first case the philosophical reasons in favor of a given mathematical ontology determine normatively how mathematics should be done; in the second case

(1) I would like to express thanks to Andrew Haigh, who helped me to improve the English version.

philosophy is a mere epiphenomenon that exerts no influence on the development of mathematics.(2)

The philosophy-oriented approach — to which Shapiro aligns himself, while insisting on the need to break down unnecessary walls between philosophy and mathematics and on the desirability of considering philosophical normativity defeasible — assumes that philosophy is an activity carried out exclusively by professional philosophers (who may possibly also be competent in mathematics) and that it influences mathematics without being in turn significantly influenced by it. To this approach we ascribe the investigations related to the development of the twentieth-century 'isms' (logicism, pragmatism, structuralism, etc.), the research related to the well-known dilemma of Benacerraf on the indispensability of mathematics, and more generally all the contributions to the problem of the existence of abstract objects, their cognitive access and the choice of a first or second order logic.

The practice-oriented approach to mathematics, on the other hand, is characterized by the idea that the search for solutions to open problems in mathematics as well as the search for definitions and arguments are essentially philosophical activities — regardless of who is in charge of them — that can have repercussions as much on the development of mathematical theories themselves as on the transformations of the conceptual tools that philosophy uses. In the practice-oriented approach we include investigations that show the intersection of philosophy and history of mathematics, epistemological research related to visualization and explanation in mathematics, investigations of computer-assisted demonstrations, and analysis of certain properties of mathematical arguments, such as purity, evidence, and fruitfulness.(3)

(2) See [Shapiro 1997, pp. 25ff]. The *philosophy-driven* and *practice-driven* distinction partly traces the opposition drawn by Paolo Mancosu between analytic philosophy and the maverick tradition, which includes [Lakatos 1976] and [Corfield 2003]. See [Mancosu 2008, p. 3].

(3) The practice-driven approach as it is understood here includes, for example, [Lakatos 1976] and [Corfield 2003], but also the new epistemology ([Mancosu 2008]), the interactions between philosophy and history ([Ferreirós and Gray 2006]; [Kerkove, De Vuyst, and Bendegem 2010]), recent investigations of argumentation in logic and mathematics ([Gabbay et al. 2002], [Aberdein and Dove 2013]), research on set theory, probability, computability, applications and open problems in mathematics ([Irvine 2009, pp. 461ff.] [Colyvan 2012]). This distinction intertwines but does not coincide with the distinctions drawn by Cellucci (2013, pp. 93-96) between static (aimed at the justification of an established body of

A relevant area of research that requires a back and forth between philosophy first and philosophy last-if-at-all approaches is the investigation of axiomatic theories and methods. Philosophy offers relevant tools for the conceptual analysis, but mathematics is the starting point since the object of the inquiry are given mathematical theories.

§ 2. — What is axiomatics?

'Axiomatics' is used in the following as a general term that stands for an inquiry into axiomatic theories and methods, including specific epistemological views and technical solutions, but also meta-theoretical considerations, and different objectives that the axiomatic method and theories might serve. Differences might concern also the purposes that the axiomatic method or the axiomatic formulation of a theory serve in a specific case (section 3). This paper will list different purposes that the axiomatic formulation of a theory might serve and illustrate them with an example taken from the investigation of the Peano School (section 5).

Let us begin with a preliminary conceptual clarification of the terms 'axiomatic method' and 'axiomatic theory'.

The 'axiomatic method' used for the formulation of mathematical theories comes in many forms, which differ by the language used: formal, informal or semi-formal. Even when the language is formal, there are several possibilities of reconstruction, for example by using first or second order logic, classical or intuitionistic logic, etc. Variations in the axiomatic method over time are well known in the literature, and a radical distinction is generally drawn between the classical Euclidean method and the modern hypothetico-deductive method. The former is considered as a content-oriented approach (*inhaltlich*), in which the axioms are considered as self-evident truths and the fundamental concepts are explicitly defined and known by intuition. The latter is presented as a formal approach, in which the axioms are hypotheses defining the primitive concepts and their mutual relations.

knowledge) and dynamic (interested rather in the growth of mathematical knowledge) philosophy of mathematics and between top-down (starting from some general unproven assumption) and bottom-up (starting from the activity of individuals) approaches. While it shares with the dynamic approach an interest in inductive arguments alongside deductive ones, it is bottom-up in the sense that it emerges in the activity of individual mathematicians.

More recent research has shown that the change in the axioms chosen to underpin a theory often reflected a profound methodological and epistemological change.(4) Besides, the very notions of axiom and definition changed meaning several times,(5) making it difficult to compare different axiomatic systems without taking into account precise historical and conceptual differences.

The notions of logical consequence and more generally of logical rule are also closely linked to the axiomatic method and have undergone profound variations not only in the transition from syllogistic to twentieth-century symbolic logic (a transition that was by no means linear, as is still evident in Peano's school),(6) but also in the transformation of the role of propositions assumed as primitives and their relation to theorems. The study of various instances of axiomatic method suggests the existence of multiple variants which are in some way intermediate between the classical (Euclidean) axiomatic method and the modern hypothetico-deductive method.

An 'axiomatic theory' is composed of a language, a logical theory — including primitive logical terms, logical axioms and logical rules of inference — a set of primitive or undefined terms, a set of statements considered as axioms, a set of defined terms and a set of theorems that can be derived from axioms by means of the logical rules of inference (often, specific rules of derivation are allowed beside the fundamental logical rules). Set theory comes in a variety of different formulations, which are related to the technical objective of avoiding paradoxes (e.g., modifications to the comprehension axioms as made in type-theory) but also to the epistemological objective of offering an adequate account of the infinite. If the different formulations of set theory reflect different philosophical viewpoints (predicativism, finitism, and so on), one can find alternative axiomatic formulations of a theory even inside a single research group: for example, in the Peano School many different formulations have been offered of the theory of natural numbers, rational numbers, real numbers, complex numbers, vector spaces, projective geometry, plane and solid geometry.

The same axiomatic theory can be analyzed from a syntactic point of view as a set of statements, from a semantic point of view as a set of non-linguistic models that satisfy the statements,

(4) See e.g., [De Risi 2016] on the role played by the parallel postulate in geometry.

(5) See e.g., [Cantù 2018] on the deductive role played by definitions in Wolff's mathematical method.

(6) [Cantù 2022b].

and finally, from a pragmatic perspective, as a set of examples, problems, norms, skills and practices. Yet, the general objective of axiomatics as a philosophical investigation of mathematical practices is not limited to the axiomatic method and the type of language and logic used, nor to axiomatic theories understood as sets of axioms and theorems or as theories composed of language, semantics, syntax and pragmatics. For example(7), axiomatics also questions the historical reasons why a certain way of conceiving the method and the axiomatic theories has been privileged in a given conceptual framework; it questions the role that axioms play in the foundations of mathematics, in the development of a mathematical style and in the institutionalization of a given social mathematical practice. The paper will thus consider three complementary ways to investigate 'axiomatics' as a whole: foundational, stylistic and institutional (section 4).

The axiomatic treatment of scientific theories was central to the early twentieth-century philosophy of logic and mathematics program and was a fundamental feature of the philosophy of science until at least logical empiricism. In more recent times, the general philosophy of science has moved toward less systematic approaches, which by undermining the unity of science at various levels have also called into question the methodological unity based on axiomatization of theories.(8)

This is one possible reason for the lack of a systematic survey of the role of axiomatics not only in contemporary philosophy of science, where it seems to have fallen out of favor, but also in the period from the late nineteenth to the mid-twentieth century. The temporal distance allows us to observe that in the first half of the twentieth century many axiomatic methods, styles, and theories were indeed developed that do not necessarily constitute a unity. The question is then whether it is possible to speak of axiomatics as a variegated complex of mathematical research and practices that can be characterized either on the basis of the common goal of investigating the foundations and the unity of science, or on the basis of some peculiar stylistic features, or even on the basis of some institutional characteristics.

Having carried out research on the axiomatization of geometry, extensive magnitudes and arithmetic, I got interested in the study of various axiomatic approaches to mathematical theories. In the

(7) [Savage 1990] ; [Winther 2016].

(8) See e.g., [Cartwright 1999].

literature devoted to the history of analytic philosophy, the role of the axiomatic method is generally associated with the study of the foundations of mathematics, the philosophical analysis of mathematical principles and rules, the discussion on the possibility of reducing mathematics to logic, and questions of purity in the relation between arithmetic and geometry. The objective here is to plea for a broader vision of what I will call ‘axiomatics’, i.e., a variety of theoretical inquiries and practical activities related to mathematics and that present different forms, objectives, and methods. Even limiting attention to the period from the late nineteenth century to the first half of the twentieth century, the variety of methods, languages, styles, practices and rules at stake in axiomatized theories is such that it suggests there is the opportunity for a wider and deeper study of the subject — an approach that is not common in the literature.(9)

§ 3. — The objectives of axiomatics.

To investigate axiomatics one needs to understand the relation between the different methods and the objectives pursued by researchers who provide axiomatic formulations of mathematical theories. In this section we will offer a (non-exhaustive) list of roles which axiomatics has played.

Heuristics. Axiomatics is useful to formulate new conjectures, to understand why some hypotheses are more important than others, to provide new demonstrations, to simplify and speed up proofs, and to discover errors.

As Hilbert says:

> However, as I have already remarked, the present work is rather a critical investigation of the principles of the euclidean geometry. In this investigation, we have taken as a guide the following fundamental principle; viz., to make the discussion of each question of such a character as to examine at the same time whether or not it is possible to answer this question by following out a previously determined method and by employing certain limited means. This fundamental rule seems to me to contain a general law and to conform to the nature of things.(10)

(9) [Schlimm 2006, pp. 2–3].
(10) [Hilbert 1902, pp. 130–131].

Freudenthal as a mathematician considers the heuristic objective of axiomatics as its main trait, and thus considers the *Grundlagen der Geometrie* as the best example of axiomatics, even if he acknowledges that an earlier formulation can be found in Padoa, and that the elimination of the link with intuition was already stated by Pasch and Fano (I would also add Bettazzi).(11) The cited example explains very well why, according to the objective considered as primary in axiomatics, one author or the other becomes the founding father.

Conceptual Analysis. Axiomatics is also a tool for conceptual analysis, i.e., a tool for understanding mathematics, and determining which concepts presuppose which, verifying whether a set of concepts are independent, and distinguishing notions expressed by a single term but having distinct logical or mathematical functions. Peano's ideography, for example, is a form of conceptual analysis(12) of the definitions occurring in mathematical textbooks. The analysis takes place in two stages that influence each other. On the one hand, the analysis of the language and of the structure of proofs allows us to highlight the primitive logical terms, for example by distinguishing the notions of membership and inclusion, often designated by the same term (the copula *est*), or by identifying the primitive logical rules (e.g., syllogism and *modus ponendo ponens*). On the other hand, the study of the mathematical definitions provided in the textbooks of analysis and geometry constitutes the starting point of the search for the fundamental concepts of mathematics and the axiomatic formulation of arithmetic, geometry, the theory of vector spaces, etc. As Peano and his collaborators clearly acknowledge, a back and forth between mathematics and logic is induced by the axiomatic treatment, because the axiomatic formulation of the mathematical theories induces a modification of the axiomatic formulation of the logical theory, and in its turn the modification of the latter requires an adjustment of the formulation of logic:

> Ideographic logic, in addition to being the most appropriate tool for a non-superficial study of logic, by abolishing any insidious promiscuity of its vocabulary with that of other sciences which presuppose only logic, like arithmetic and geometry, has obliged each of these to an equally diligent revision of their own vocabulary.(13)

(11)See [Freudenthal 1957, p. 153], [Padoa 1901], [Pasch 1884], [Fano 1958] and [Bettazzi 1890].

(12)[Schlimm 2021].

(13)See [Padoa 1933, pp. 75–77].

Quest for rigor. Axiomatics serves to eliminate the risk of internal contradictions in a theory and to eliminate the errors which could result from an excessive recourse to intuitive ideas, to the assumption of contradictory axioms, the choice of a wrong definition, the application of an incorrect rule in a derivation, or the making of a mistake in a proof. The quest for rigor can be expressed in various ways in mathematics, and it is important to distinguish between mathematical and philosophical needs. For Peano, although his position on this point differs significantly from Pieri and Padoa, the search for rigor is dictated by a conceptual analysis that is intrinsic to mathematics and not by the effort to provide a philosophical foundation for mathematics. For Bolzano, the search for rigor is associated with the task of providing the right order of presentation of the concepts and propositions of a scientific theory.(14)

The search for rigor cannot be identified neither with the conceptual analysis which allows us to identify the elements of a theory, nor with the descriptive objective consisting in the production of an error-free presentation of a theory. Indeed, there are a variety of methods to increase rigor in mathematics. In Peano's case, the method by counterexamples is the most important and is mainly applied to mathematical definitions, whose inadequacy is shown by finding examples that satisfy the general definition but should be excluded, or examples that do not satisfy the definition but are intuitively taken to fall under the given concept. For example, Peano criticizes the definition of Peano-Schwartz because it holds for the length of a curve arc but not for all concave surfaces.(15)

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(14)See [Bolzano 1804] and [Cantù 2014].

(15)Serret had defined the length of a curve arc by analogy to the length of a surface: the former being defined as the common value to the upper bound of inscribed polygons and the lower bound of circumscribed polygons, the latter was defined as the limit of an inscribed polyhedric surface. But Serret's procedure was flawed, as both Peano and Schwarz acknowledged ([Peano 1890, p. 55]). Hermite offered an alternative definition, based on the limit of a series of non contiguous polygons that are tangent to the surface, but he thus lost the analogy between the inscribed polygons and the inscribed polyhedric surfaces. Peano in 1890 suggested a new definition based on the notions of vector and bivector, which respects the analogy: the length of the curve arc is the superior limit of the sum of the vectors of its parts; the area of a surface is the superior limit of the sum of the bivectors of its parts. [Peano 1890, p. 56]. For an exhaustive analysis of the problem of the definition of the surface area, see [Gandon and Perrin 2009].

Even if we limit ourselves to the logico-philosophical field, it is worth noting that the search for rigor does not necessarily presuppose a formal conception of axiomatics, but is perfectly compatible with a content-oriented vision, as well as with a positive evaluation of the role of intuition in mathematics, which is often taken to be necessary in order to justify the applicability of mathematics.(16)

Descriptive aim. Axiomatics is used to present a theory once it has already been developed. This objective is often considered by philosophers as purely accessory, or at least secondary to conceptual analysis or search for rigor, which are usually taken as the two pillars of the axiomatic method in the philosophical investigation of mathematical foundations. However, in the history of logic, long considered as a general theory of concepts, the primary objective of the axiomatic method was that of providing a good presentation of a given topic in a treatise. The right order of concepts was thus mainly understood as the better way to introduce a topic. For example, Bolzano defined logic as the art of presenting a theory in an appropriate textbook (*Wissenschaftslehre*), which explains the kind of criticism he made to the order of the geometrical concepts as exposed in Euclid's *Elements*.(17) Often, various orders of concepts were considered, and the order of concepts present in the divine mind (*ordo essendi*) was contrasted with the order reconstructed by humans and necessary to their understanding (*ordo cognoscendi*). The difficulty of providing an exhaustive list of the fundamental concepts of a discipline was precisely linked to human finitude, a theme that we also find in Gödel, for example when he observes that the paradoxes of set theory are due to a bad choice of the axioms and not to a problem concerning the existence of sets.(18)

For philosophers of science, the descriptive objective attests to the value of axiomatics in justification contexts, and explains why it does not allow new discoveries. Hempel observes for example that axiomatics, being a 'device of exposition' of theories, is used to compare and justify them, but not to discover new facts.(19) The descriptive objective is here opposed to the heuristic objective.

(16) See for example [Gödel 1953/9, pp. 348–9].
(17) See [Bolzano 1837], [Bolzano 1804] and [Bolzano 1810].
(18) [Crocco et al. 2020, pp. 48, 73, 86].
(19) [Hempel 1970].

Genetic-historical aim. Axiomatics is sometimes used to compare different theories, to explain the transformation or the evolution of a theory into another and possibly the ensuing cognitive progress. A well-known example is Hilbert's *Grundlagen der Geometrie*, in which the distinction between groups of axioms is also used to compare different geometrical theories.(20) But I would also like to mention the case of Giuseppe Veronese, because his axiomatic formulation of infinitesimal geometry was from the beginning linked to the comparison between the non-Archimedean continuum and the continuum of Dedekind, and prompted him to write a long appendix to his book *Fondamenti della geometria* which constitutes one of the first examples of the history of mathematics being deeply influenced by axiomatics.(21)

Other interesting examples are Peano's *Formulario* and Bourbaki's *Éléments de mathématiques*, both of which are extremely attentive to the axiomatic formulations and their history, explaining the first occurrence of a certain formula or the evolution of mathematical ideas.(22)

Architectural aim. Axiomatics can be used to restructure and better understand the global edifice of a science, which evolves according to the individual axiomatic theories of which it is composed, and is renewed thanks to new structural analogies allowed by the development of axiomatics itself (think for instance of the mother structures in [Bourbaki 1950] or the disciplinary reorganization that transformed geometry from a mathematical science to a physical science in the 19th century).(23)

The reorganization of the architecture of mathematics can also influence its position in the classification of sciences, a problem that was crucial at the end of the 19th and the beginning of the 20th century, because of the multiplication of mathematical disciplines, including arithmetic, geometry and analysis, but also probabilities, combinatorics vector spaces and hypercomplex systems, and the fragmentation of other scientific disciplines, which increased the number of possible applications.

(20) [Hilbert 1899].
(21) [Veronese 1891].
(22) See [Peano 1901], [Bourbaki 1939-1984] and [Dieudonné 1978].
(23) [Torretti 1978].

Modular analysis. In some cases, the interest in a mathematical problem or theorem is not only related to the understanding of the role it plays in a given axiomatic system. What one wants to study is the amount of mathematics needed to derive the theorem or to solve the problem. An example of this is the interest for axiomatics developed by Hilbert in the *Grundlagen der Geometrie*, as he investigated what can be proved using only one or some of the five groups of axioms he isolated (incidence, order, congruence, parallelism and continuity).(24) This is the objective of reverse mathematics, to determine which axioms are necessary to prove a certain set of theorems or which set of theorems or which formal systems isolate the principles necessary to prove them.(25)

Pedagogical aim. Axiomatics can also be a pedagogical tool, not only because it allows a clear exposition of theories, but also because it develops students' abstraction skills.(26) Asking students to define a concept or prove a mathematical theorem leads to the study of several possible definitions and proofs, each based on distinct notions and rules.(27)

The effort to check that the mathematical content presented by the teacher is actually accessible to students at a certain level of study requires a distinction between elementary and non-elementary parts of a theory, and sometimes suggested alternative axiomatic formulations of a theory using only the axioms that are considered to be elementary. For example, Giuseppe Veronese distinguished an intuitive, experimental and practical method of teaching, which he considered adequate for young children, from a rational and rigorous axiomatic method, which he recommended not to introduce before the age of 14. The geometrical propositions taught in the two cases are not the same, because material or practical geometry is limited to the field of the observable: instead of stating geometric propositions about unlimited lines, he referred only to line segments, so as to include only propositions that can be experimentally verified.(28)

The pedagogical objective is often associated with a descriptive objective, because in the writing of a textbook the right order of

(24) [Hilbert 1899].
(25) [Friedman 1975, p. 235].
(26) [Piaget 1968].
(27) [Peano 1921].
(28) [Veronese 1909].

concepts is the one that facilitates understanding without requiring concepts that are foreign to the learners' field of knowledge. The problem of purity — i.e., the search for a proof of a theorem that does not call upon notions foreign to the theory in question — originated in the philosophical concern to adhere to the Aristotelian prohibition rule on kind crossing: e.g., one can forbid the use of arithmetical tools in a geometrical proof or theorems of spherical geometry in plane geometry. But it also emerged in the didactic concern of presenting a proof that could also be made by students fully autonomously.

Coordination tool. If mathematics is considered as a social activity carried out by actual human agents or multi-agent systems that find themselves with the practical necessity to regulate their interactions, an axiomatic theory can be used to solve the problems of coordination.

This problem did not originate neither in the recent tendency of the philosophy of mathematics to deal with concrete scientific practices, nor in computer-assisted proofs. Peano already envisaged the urge to coordinate mathematical activity, as he justified the wish to build a collection of mathematical formulas by the need of a practical tool to distinguish what has already been demonstrated from what is still in need of a proof.(29)

Coordination problems can also be solved through a genetic-historical approach, which allows us to compare theories developed at different times and with different tools, and thus also to understand to what extent a new proof provided with different techniques can be more or less adequate or general than a pre-existing proof.

From the perspective of the external history of mathematics, which focuses on concrete institutions in which mathematical agents operate (universities, journals, research institutions, schools, etc.), the objective of coordination becomes essential to understanding the dynamics of collective endeavors, peer review and scientific controversies.

To the extent that the unification of mathematics or science is also part of the instruments that promote the coordination of mathematical agents, this objective also intersects the architectural goal mentioned above.

(29) [Peano 1896].

§ 4. — Three conceptions of axiomatics in mathematics.

Axiomatics is generally investigated as a method that facilitates research on the foundations of mathematics. But recent developments in historical epistemology and social ontology might suggest alternative or complementary viewpoints: axiomatics can be considered as research on the foundations of mathematics, as a mathematical style, or as a social institution.

Axiomatics as research on the foundations of mathematics. Axiomatic systems and methods have often been introduced to deal with the question of the foundations of mathematics, that is to say, to determine what the principles of science are, by possibly distinguishing *the ordo essendi* from *the ordo cognoscendi*, i.e., what is first in itself from what is first for us. In this classical sense, originated in Aristotle's *Prior Analytics*, axiomatics has an ontological and cognitive scope. Key questions posed by axiomatics are reductionism (which principles are really independent of each other and which are those that can be reduced to principles of other disciplines or to logical principles?) and purity (is it possible to formulate an axiomatic theory without incorporating principles from other disciplines or sciences?). Within this general framework, different interpretations of the question of foundations (relating, for example, to rigor and the elimination of errors in mathematical definitions and proofs, or to the philosophical clarification of the nature of fundamental elements) and of formal systems as closed or open, lead to an enormous variety of axiomatic formulations of a theory.

Actually, there might be two distinct ways to approach the foundations of mathematics. By mathematical problem of the foundations, I mean here the objective to give a rigorous organization to the whole of mathematics, by providing a precise and unambiguous characterization of its key concepts (e.g., limit and continuity in the case of analysis).(30) Among the authors who have contributed to the renewal of mathematics in this sense I include Gauss, Abel, Cauchy, Bolzano, and in the historical phase of the arithmetization of the analysis, Weierstrass, Cantor and Dedekind (but also Kossak, Meray, Heine, Lipschitz and Tannery), who tried to avoid the use of the notion of geometric continuity in the definition of the properties of real numbers.

By logico-philosophical foundation of mathematics, I rather mean a reflection on the nature of symbolism and abstraction used in mathematical practices, on the conditions that legitimize their

(30) [Mangione and Bozzi 1993, p. 269].

application and justification.[31] The fundamental question is associated with a hypothetico-deductive view of axiomatic theories and to the discussion of alternative philosophical points of view on mathematics: logicism, intuitionism and formalism.[32]

A similar distinction is made by Leo Horsten: the inquiry is called *foundational research* when the mathematicians themselves are "concerned with the foundations of their subject" and *philosophy of mathematics*, when "philosophers investigate philosophical questions concerning mathematics".[33]

Axiomatics is one of those areas that requires a joint interaction of the two approaches, so as to avoid philosophy of mathematics being reduced to a list of 'isms' that investigate the nature of mathematical entities or how we can have knowledge of them. Axiomatics (generally a Hilbert-style axiomatics) is often taken to be the general framework in which these discussions and analyses are made. It is seldom questioned in the philosophy of mathematics itself, but rather delegated to philosophy of logic.[34]

Axiomatics is also a general topic of philosophy of science: a unitarian view, guided by a predominantly ontological interest, is the Classical Model of Science, which presents a list of relevant features of the method that are considered as still influential in contemporary science.[35] The consideration of the mode of presentation of a set of propositions and how this architecture might be influential in forming the epistemic attitude of the reader is seldom discussed, but this was a central feature of logic in Bolzano and more generally in the 19th century. An exception is constituted by researchers who looked for a joint answer to philosophical and historical questions, thereby investigating axiomatics both as a cultural phenomenon and a foundational issue.[36]

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(31) [Mangione and Bozzi 1993, p. 262].

(32) [Shapiro 2005].

(33) See [Horsten 2022]. Note that philosophy of mathematics need not be exclusively identified with the investigation on the foundations of mathematics, as has been argued by philosophers of mathematical practices as well as by historians of mathematics. I will not enter in the discussion here, but the reader might find surveys of these approaches in [Carter 2019; Ferreirós and Gray 2006; Giardino 2017; Kerkove, De Vuyst, and Bendegem 2010; Mancosu 2008].

(34) See e.g., [Jacquette 2002]. But note that [Quine 1970] claimed that axiomatics does not have an impact on logical theory, because the latter is a set of logical truths, no matter which are chosen as axioms and which are derived from them.

(35) [Jong and Betti 2010].

(36) See e.g., [Ferreirós and Gray 2006].

Axiomatics as a mathematical style. When axiomatics is considered as a cultural feature that is embodied in a particular mathematical theory, it is legitimate to wonder whether it could be described as a style.(37) The term 'style' has been introduced in mathematics by analogy with artistic or literary styles, and has been brought into the limelight by different traditions. Chevalley, a member of the Bourbaki group, remarked that axiomatics has deeply modified the style of contemporary mathematical writings.(38) Granger developed a general study of style in mathematics, philosophy of language and human sciences.(39)

Crombie gave an ostensive explanation, individuating six fundamental styles that characterize the scientific enterprise: postulation (Greek mathematics), experiment, hypothetical construction of analogical models, ordering by composition and taxonomy, statistical analysis and probability, and historical derivation of genetic development. Hacking required two further conditions for the identification of a new style: novelty and persistence through self-stabilization. Historical epistemology also insisted on the cultural features of styles, as results of the activity of a specific school, nation or tradition.

Paolo Mancosu, comparing the use of the term made in history of mathematics and in mathematics, distinguished several kinds of style: individual, methodological, epistemic, national, cultural, writing, thinking or cognitive style.(40)

Nowhere is axiomatics characterized as a style, even if Crombie's characterization of the postulationist style of Greek mathematics goes in this direction and Chevalley highlighted the relation between axiomatics and style modifications.

The question here is how to characterize axiomatics as a style. The term might refer to the writing style of a certain mathematician, as when one speaks of axiomatics à la Dedekind or à la Peano, or, as is often done, of Hilbert-style axiomatics. In the latter case, an individual axiomatic style is also a methodological style originating in a research tradition: as Chevalley remarks, the ϵ-style

(37) This idea first came to me while reading [Mancosu 2017], [Marquis 2022], [Rabouin 2017], and was supported by fruitful discussions with Frédéric Patras and Sébastien Maronne on Bourbaki.
(38) [Chevalley 1935, p. 375].
(39) [Granger 1988].
(40) [Mancosu 2017].

was an author's style that became specific to a mathematical era.(41) The same holds for Buss's distinction between proof theory in the Hilbert style, in which each step of a derivation is a formula, and proof theory in the Gentzen style, in which each step of a derivation is a sequent.(42)

In another sense, axiomatics can be considered as an element of a style of thinking, i.e., as a component of a paradigm or as an epistemic concept.(43) Axiomatics can still be interpreted as a cognitive style, be it the structure of human psychological development(44) or one of the many cognitive styles preferred by agents in teaching and learning.(45)

In a historical context, it is possible to consider the axiomatic style as a historiographical or mathematical category — think of the contrast between the deductive Aristotelian conception of mathematics, in which axioms are true and self-evident, and the Hilbertian hypothetico-deductive conception(46) — or as an individual style, illustrated for example by Zermelo in the case of set theory.(47)

The axiomatic style could then give rise to sub-styles. The distinction between (a) a hypothetico-deductive approach, which considers a theory as a closed system, (b) a semantic approach, which sees a theory as a set of models, and (c) an analytical approach — compatible with the understanding of a theory as an open system(48) — could be seen as a difference in axiomatic style.

Two other opposite forms of axiomatic style would be content-oriented axiomatics, which includes Euclid's geometry, Newton's mechanics and Clausius' thermodynamics, and formal or existential axiomatics.(49) Similarly, one could compare the style of the modernists, for whom mathematics has only to do with words, with the style of the anti-modernists, for whom mathematics has to do with objects.(50)

(41) [Chevalley 1935].
(42) [Buss 1998].
(43) See respectively [Kuhn 1962] and [Hacking 1999].
(44) For [Piaget 1947] the mental development of the child is associated with the progressive unraveling of logical-mathematical axioms.
(45) See [Borromeo Ferri 2005] and [Lerman 1990].
(46) [Kline 1990, vol. 3]
(47) See [Zermelo 1908] and [Gray 2008, p. 260]. See also [Lorenzo 1971].
(48) [Cellucci 2017].
(49) See [Hilbert and Bernays 1934, p. 2] and [Sieg and Ravaglia 2005, p. 987].
(50) See [Mehrtens 1990] and [Gray 2008, p. 9].

Axiomatics as an institution. A third way of approaching axiomatics is to consider its persistence in time, although in changing forms, as a cognitive enterprise realized by human agents to regulate complex interactions. The possibility to distinguish axiomatics as a *type* from its instantiations (*tokens*) based on different methods and different formulations of theories, and the presence of rules, functions agents and roles, legitimates the question of whether axiomatics can be considered as a social institution. In social ontology, the philosophical discipline that studies social institutions and their components, an institution is characterized by rules, obligations, coordination problems, agents and roles.(51)

Formal axiomatics could be associated with the following set of rules: 1) determine the primitive terms of a theory and the axioms that implicitly define them; 2) make the legitimate rules of derivation explicit; and 3) specify which rule is applied at each stage of a proof. In this approach, some of the classical meta-theoretical problems associated with axiomatics could be considered as specific *obligations*. For example, consistency would be an obligation justified by the fact that it is undesirable that a statement and its negation are both axioms or theorems of the same system. Completeness in the Hilbertian (geometrical) sense could be considered as an obligation dictated by the desire that the axiomatic system has an adequate level of generality. Agents include human beings, computers and multi-agent systems. Roles include researcher, teacher, student, newspaper reporter, etc. Many coordination problems could be facilitated by axiomatics: *a*) avoid proving a theorem that has already been satisfactorily proven; *b*) provide a common conceptual framework for comparing different proofs of the same theorem and evaluate which one is the most adequate; *c*) facilitate the interaction and coordination between different researchers, groups of researchers and research projects in proving new results or formulating new conjectures; *d*) facilitate the unification of fragmented results into an organic unity.(52)

(51) [Epstein 2016; Guala 2016].

(52) This interpretation of axiomatics as a social institution was the subject of a presentation at the ENPOSS 2020 international symposium.

§ 5. – Peano's example.

Peano developed axiomatics as a tool for the scientific study of the foundations of mathematics, i.e., for organizing the corpus of mathematical knowledge and providing a precise and adequate characterization of its key terms. Peano also dealt with the logico-philosophical foundations of mathematics, especially in his reflections on the nature of symbolism and abstraction used in mathematical practice, but without adhering to any form of philosophical foundationalism, since the search for rigor is based neither on rigid deductivism nor on any form of reductionism.

Peano developed a specific axiomatic style, based on the method of counterexamples applied to definitions in order to find the right level of generality (for example, providing a definition of the length of the curvilinear arc applicable to all concave surfaces), a style based on the refusal to lean clearly towards an exclusively extensional or exclusively intensional interpretation of logic and mathematics, a style based on a particular way of associating syntax, semantics, and pragmatics through the grammar of language.(53)

Finally, Peano institutionalized axiomatics through collaborative projects: the *Formulario*, the *Dizionario*, the *Revue des Mathématiques* and the publications of the *Academia pro Interlingua*. The journals were the instruments for coordinating the work, but participation in national and international conferences also provided an opportunity to invite contributions, critiques, and corrections to collaborative projects: e.g., the conference held in Livorno in 1901 launched the project for a dictionary of mathematics.

The institutional nature of axiomatics is not only the result of an intersubjective process, but is also shaped by the way interactions are regulated and certain tasks are assigned to agents. As can be seen from Enriques' reading of Peano, axiomatics can be seen as a set of four rules: 1) all concepts must be taken as primitives or logically derived from primitives; 2) primitive concepts must be independent; 3) all propositions must be taken as axioms or derived from axioms; 4) axioms must be independent.(54)

Each of these perspectives on axiomatics (inquiry into foundations, mathematical style, social institution) can be characterized

(53) See [Cantù 2022a] and [Luciano 2017].

(54) [Enriques 1924–27, pp. 11-12].

by the study of the objectives it is called upon to fulfill, which, in Peano's case, are indeed multiple.

1. Axiomatics is essential to conceptual analysis, understood as a preliminary operation to the introduction of an ideography.
2. It serves a heuristic objective, facilitating the search for errors in the definitions, but also the search for new examples and problems.
3. It induces a quest for rigor, notably elimination of inadequate definitions and sophisms.
4. It has the descriptive capacity to represent contemporary mathematical practices.
5. Axiomatics also produces a history of notations and theorems.
6. It allows a comparison of alternative theories on the basis of criteria of simplicity.
7. It facilitates an architectural organization of mathematical knowledge.
8. Axiomatics also plays the role of a modular analysis, either of the theorems derivable from a single axiom, or of the axioms necessary to derive a given theorem;
9. It also has a didactic objective, aimed at clearly explaining the theories and developing the students' abstraction capacities;
10. Finally, axiomatics also plays the role of a coordination tool allowing mathematicians to quickly discover whether a certain proposition has already been proved and with what resources.

§ 6. — Conclusion.

This article has attempted to show how philosophy, even if not taken as a normative starting point of the inquiry, can contribute to a reflection on the axiomatic activity of mathematicians. It is often said that mathematicians do not work axiomatically, and that this method is mainly concerned with organization rather than scientific discovery. Here we have tried to show that axiomatics, if analyzed in detail through a study of its foundational component, of the styles with which it is associated and of the rules that govern it, performs a plurality of functions. But this analysis requires an interdisciplinary approach in which mathematics, philosophy and history of logic and mathematics are involved: conceptual analysis cannot be dissociated from the punctual study of mathematical theories and their historical development. The analysis of Peano's example offers insight into the kind of results that this interdisciplinary approach to mathematical practice might produce.

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