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Introducing Heuristic Philosophy of Mathematics

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Abstract. Mainstream philosophy of mathematics, namely the philosophy of mathematics that has prevailed for the past century, holds that mathematics is theorem proving by the axiomatic method. But this is incompatible with Gödel's incompleteness theorems, and cannot account for many features of mathematics. This article proposes an alternative approach, heuristic philosophy of mathematics, according to which mathematics is problem-solving by the analytic method. The article argues that this is compatible with Gödel's incompleteness theorems, and can account for the features of mathematics not accounted for by mainstream philosophy of mathematics, such as the nature of mathematical objects and mathematical definitions.

Keywords. Mainstream philosophy of mathematics • Mathematics as theorem proving • Heuristic philosophy of mathematics • Mathematics as problem-solving • Philosophy of mathematical practice • Mathematical objects • Mathematical definitions

§ 1. — Introduction.

The purpose of this article is to outline the main aspects of heuristic philosophy of mathematics.

Heuristic philosophy of mathematics is an approach to the philosophy of mathematics alternative to mainstream philosophy of mathematics, the philosophy of mathematics that has prevailed for the past century

Mainstream philosophy of mathematics consists of the three big foundational schools (logicism, formalism, intuitionism), their direct descendants (neo-logicism, neo-formalism, neo-intuitionism), and their indirect descendants (platonism, structuralism, fictionalism, nominalism, etc.).

The three big foundational schools and their direct or indirect descendants are different in several respects but have certain characteristics in common. These characteristics may serve to characterize mainstream philosophy of mathematics.

§ 2. — Mainstream Philosophy of Mathematics.

The main characteristics of mainstream philosophy of mathematics are the following.

(1) The philosophy of mathematics cannot concern itself with the making of mathematics, in particular discovery, because discovery is a subjective process, so it cannot be justified.

(2) The philosophy of mathematics can concern itself only with finished mathematics, namely mathematics as presented in finished form in journals, books, or lectures, because only finished mathematics is objective, so it can be justified.

(3) Since the philosophy of mathematics cannot concern itself with the making of mathematics, it cannot contribute to the advancement of mathematics.

(4) The task of the philosophy of mathematics is primarily to give an answer to the question: How do mathematical propositions come to be completely justified? And, subordinately to it, to the question: Do objects exist in virtue of which mathematical propositions are true (in some sense of 'true')? And, if such objects exist, what is their nature?

(5) The method of mathematics is the axiomatic method. The latter is the method according to which, to obtain a demonstration

of a proposition, one starts from given axioms, which are true (in some sense of 'true'), and deduces the proposition from them.

(6) The role of demonstration is to guarantee the truth of a proposition.

(7) Mathematics is a body of truths, and indeed truths that are certain. Therefore, mathematics is about truth and certainty.

§ 3. — Mainstream Philosophy of Mathematics and Closed Systems.

From the above description of the main characteristics of mainstream philosophy of mathematics, it appears that a basic assumption of mainstream philosophy of mathematics is that mathematics is theorem proving by the axiomatic method.

The basic assumption implies that mathematical theories are closed systems. A closed system is a system whose development does not involve receiving inputs from and delivering outputs to the outside. Its development remains completely internal to the system, so the system is a self-sufficient totality.

The basic assumption implies that mathematical theories are closed systems because a mathematical theory is based on axioms that are given once for all, and its development consists entirely in deducing propositions from the axioms.

Now, deduction is non-ampliative, namely the conclusion contains nothing essentially new with respect to the premisses. For, as Russell says, a deductive inference "consists merely in saying in other words part or the whole of what is said in the premisses" (Russell 1997, 360).

That deduction is non-ampliative has been repeatedly pointed out from antiquity. Thus, Epicurus says that "we will not say that syllogism can make us know anything new" (Epicurus, *De natura*, XXVIII, 17, ɪ, ed. Sedley). Descartes says that logicians cannot "form any syllogism leading to a true conclusion unless" they "already knew the very same truth which is deduced in the syllogism," so they "can learn nothing new from such form of reasoning" (Descartes 1996, X, 406). Kant says that "using deduction as a tool for an expansion of information comes down to nothing but idle chatter" (Kant 1998, A61/B86). De Morgan says that deduction is subject "to the great rule of all search after truth, that nothing is to be asserted as a conclusion, more than is actually contained in the premises" (De Morgan 1835, 99). Mill says that, in the conclusion of a deduction, there is "nothing but what was already asserted in the premisses" (Mill 1963–1986, VII, 160). Peirce says that deduction "is evidently entirely inadequate to" go "beyond the facts given in the premisses" (Peirce 1931–1958, 2.681). Poincaré says that deduction is "incapable of adding anything to the data" which are "given it; these data reduce themselves" to "axioms, and we should find nothing else in the conclusions" (Poincaré 2015, 31). Wittgenstein says that "if one proposition follows from another, then the latter says more than the former, and the former says less than the latter" (Wittgenstein 2002, 5.14).

Since deduction is non-ampliative, the propositions deduced from the axioms are already implicitly contained in the axioms. Therefore, the development of a mathematical theory remains completely internal to the theory, it does not involve interactions with other mathematical theories.

As Hintikka says, in the axiomatic method you compress "all the truths about" a given "subject matter into" a "set of axioms," which "are supposed to tell you everything there is to be told about this subject matter" (Hintikka 1998, 1). Then "the rest of your work will consist in merely teasing out the logical consequences of the axioms. It is sufficient to study the axioms" (ibid., 2).

§ 4. — Mainstream Philosophy of Mathematics and Gödel's Incompleteness Theorems.

The basic assumption of mainstream philosophy of mathematics, that mathematics is theorem proving by the axiomatic method, is refuted by Gödel's incompleteness theorems.

The basic assumption is refuted by Gödel's first incompleteness theorem because, by the latter, for any consistent, sufficiently strong, formal system, there are propositions of the system that are true but cannot be deduced from the axioms of the system. This implies that mathematics cannot consist in the deduction of propositions from given axioms because, for any choice of axioms for a given part of mathematics, there will always be true propositions of that part which cannot be deduced from the axioms.

The basic assumption is also refuted by Gödel's second incompleteness theorem because, by the latter, for any consistent, sufficiently strong, formal system, it is impossible to demonstrate, by absolutely reliable means, that the axioms of the system are consistent. So, there is no guarantee that the propositions deduced from the axioms are justified knowledge, because from inconsistent axioms all propositions, even logically contradictory ones, can be deduced.

§ 5. — Other Shortcomings of Mainstream Philosophy of Mathematics.

In addition to the shortcoming shown by Gödel's incompleteness theorems, mainstream philosophy of mathematics has other shortcomings.

(1) Mainstream philosophy of mathematics does not account for the fact that solving a problem of a given part of mathematics may require hypotheses from other parts of mathematics.

For example, to solve the problem posed by Fermat's conjecture, which is a problem about the integers, Ribet used a hypothesis about modular forms in hyperbolic space, the Taniyama-Shimura conjecture: Every elliptic curve over the rational numbers is modular. Then, to solve the problem posed by the Taniyama-Shimura conjecture, Wiles and Taylor used hypotheses from various parts of mathematics, from differential geometry to complex analysis.

Mainstream philosophy of mathematics does not account for this fact because, according to it, a solution to a problem of a given part of mathematics should be deduced from the axioms for that part.

(2) Mainstream philosophy of mathematics does not account for the fact that a demonstration yields something new.

For, according to it, a solution to a problem is deduced from the axioms. Now, a deduction from the axioms cannot yield anything essentially new with respect to them because, as already said, deductive rules are non-ampliative.

(3) Mainstream philosophy of mathematics does not account for the fact that new solutions, even hundreds of them, are often sought for problems for which a solution is already known.

In fact, giving new demonstrations of result already demonstrated has been a salient feature of mathematics since antiquity. As Knorr says, "multiple proofs were frequently characteristic of pre-Euclidean studies" (Knorr 1975, 9). But they have been frequently characteristic of post-Euclidean studies as well. For example, for the Pythagorean theorem over four hundred demonstrations are known, and their number is still growing. As another example, a Fields Medal was awarded to Selberg (among other important results) for producing a new demonstration of a theorem, the prime-number theorem, for which a demonstration was already known. For several examples of theorems with multiple demonstrations, see Dawson (2015).

Mainstream philosophy of mathematics does not account for this fact because, according to it, a demonstration establishes the truth of a theorem, and this is its function. Then, once the truth of a theorem has been established, there is no point in establishing it once again by another demonstration, let alone by hundreds of them.

Wittgenstein says: "Every proof, even of a proposition which has already been proved, is a contribution to mathematics". But "why is it a contribution if its only point was to prove the proposition?" (Wittgenstein 1978, III, § 60). Mainstream philosophy of mathematics cannot answer this question.

Heuristic philosophy of mathematics aims to remedy the shortcomings of mainstream philosophy of mathematics.

§ 6. — Heuristic Philosophy of Mathematics.

The main characters of heuristic philosophy of mathematics are the following.

(1) The philosophy of mathematics can concern itself with the making of mathematics, in particular discovery, because discovery is an objective process, so it can be accounted for.

(2) The philosophy of mathematics can concern itself also with finished mathematics. But finished mathematics is never really finished, because every mathematical concept or hypothesis can always be called into question, modified, or reinterpreted.

(3) Since the philosophy of mathematics can concern itself with the making of mathematics, it can possibly contribute to the advancement of mathematics. In fact, from antiquity, the philosophy of mathematics has repeatedly contributed to it. For example, as Grabiner says, Berkeley's attack on the calculus "pointed out real deficiencies" and indicated "the questions which had to be answered if a successful foundation were to be given" (Grabiner 2005, 27).

(4) The task of the philosophy of mathematics is primarily to give an answer to the question: How is mathematics made? And,

subordinately to it, to the questions: What is the nature of mathematical objects, demonstrations, definitions, diagrams, notations, explanations, beauty, applicability, and knowledge?

(5) The method of mathematics is the analytic method. The latter is the method according to which, to solve a problem, one looks for some hypothesis that is a sufficient condition for solving the problem, namely such that a solution to the problem can be deduced from the hypothesis. The hypothesis is obtained from the problem, and possibly other data, by some non-deductive rule. The hypothesis must be plausible, namely the arguments for the hypothesis must be stronger than the arguments against it, on the basis of experience. But the hypothesis is in turn a problem that must be solved, and is solved in the same way. Namely, one looks for another hypothesis that is a sufficient condition for solving the problem posed by the previous hypothesis. The new hypothesis is obtained from the previous hypothesis, and possibly other data, by some non-deductive rule, and must be plausible. And so on. Thus, solving a problem is an ongoing process. The analytic method dates back to the beginnings of mathematics. Its first documented uses are the solutions of Hippocrates of Chios to the problems of doubling the cube and the quadrature of certain lunules. For them, and the analytic method generally, I refer the interested reader to Cellucci 2022, chap. 5.

(6) The role of demonstration is to discover a plausible solution to a problem.

(7) Mathematics is a body of problems and plausible solutions to them. Therefore, mathematics is not about truth and certainty, but about plausibility.

§ 7. — **Origin of Heuristic Philosophy of Mathematics.**

The origin of heuristic philosophy of mathematics can be credited to Lakatos.

Indeed, Lakatos criticizes mainstream philosophy of mathematics because in it "there is no proper place for methodology qua logic of discovery" (Lakatos 1976, 3). Mainstream philosophy of mathematics "denies the status of mathematics to most of what has been commonly understood to be mathematics," in particular it can say nothing about "the 'creative' periods" and "the 'critical' periods of mathematical theories" (ibid., 2). On the contrary, the philosophy of mathematics must be concerned with methodology qua logic of discovery. Although there is no infallibilist logic of discovery, namely "one which would infallibly lead to results, there is a fallibilist logic of discovery" (ibid., 143–144, footnote 2). The latter consists in "the method of proof and refutations" (ibid., 50). According to it, a mathematician discovers solution to problems "by trial and error" (ibid., 73).

Lakatos's approach, however, has an important shortcoming. Lakatos assumes that the logic of discovery consists in the method of proof and refutations, according to which a mathematician discovers solutions to problems by trial and error. The assumption is untenable, because the number of trials a mathematician can make is very small with respect to all possible ones, so the probability that a mathematician may discover solutions to problems by trial and error is very low. This contrasts with the fact that over 100,000 research papers in mathematics are published every year.

Lakatos himself ends up admitting that the method of proof and refutations does not provide a basis for methodology qua logic of discovery. Indeed, he says that "modern methodologies or 'logics of discovery' consist merely of a set" of "rules for the appraisal of ready, articulated theories" (Lakatos 1978, I, 103). These rules do not "give advice" as to "how to arrive at good theories" (Lakatos 1971, 174). They only give "directions for the appraisal of solutions already there" (Lakatos 1978, I, 103, footnote 1).

Now, as Nickles says, it "is astonishing" that "Lakatos's methodology provides ways to appraise" solutions already there, "but stops short of giving advice" (Nickles 1987, 119). For, "the idea of a heuristic methodology which gives no advice is a contradiction in terms. Bluntly stated, Lakatos has no methodology" (ibid.,120).

This does not invalidate the claim that the origin of heuristic philosophy of mathematics can be credited to Lakatos. But it means that, with respect to heuristic philosophy of mathematics, Lakatos is a sort of 'non-playing captain', namely a captain who is not in the field when the game takes place.

Heuristic philosophy of mathematics, as formulated here, is not subject to the shortcoming of Lakatos's approach concerning the method of proof and refutations. For, it replaces the method of proof and refutations with the analytic method, in which hypotheses are discovered, not by trial and error, but by non-deductive inferences.

§ 8. — Other Objections to Heuristic Philosophy of Mathematics.

In addition to the objection concerning Lakatos's method of proof and refutations, other objections have been raised against Lakatos. They, however, are invalid. I will consider them because such objections could also be raised against heuristic philosophy of mathematics as formulated here, therefore, it is important to show that they are invalid.

(1) Feferman says: "Lakatos' fireworks briefly illuminate limited portions of mathematics," but only deductive "logic gives us a coherent picture of mathematics," it "alone throws light on what is distinctive about mathematics, its concepts and methods" (Feferman 1998, 93). The "logical analysis of the structure of mathematics has been especially successful," one can use formal systems also "to model the growth of mathematics" (ibid., 92).

But this objection is invalid. By the strong incompleteness theorem for second-order logic, there is no consistent formal system for second-order logic capable of deducing all second-order logical consequences of any given set of propositions. Now, much of mathematics requires second-order logic or beyond. Therefore, deductive logic cannot be said to give us a coherent picture of mathematics, nor to throw light on what is distinctive about mathematics, its concepts and methods. Moreover, deductive logic is non-ampliative, so it cannot explain why a demonstration may yield something new. Therefore, formal systems cannot be used to model the growth of mathematical knowledge.

(2) Smoryński says: "Lakatos firmly denies the distinction between mathematics, on the one hand, and the "sciences on the other," but "I cannot accept such a denial" (Smoryński 1983, 11). Mathematics and the sciences are essentially different because, in the sciences there are revolutions, while mathematics "is a cumulative body of knowledge" (ibid.).

But this objection is invalid. Mathematics is not a cumulative body of knowledge, there are revolutions also in mathematics. This depends on the fact that in mathematics we start from problems, we formulate hypotheses for their solution by non-deductive inferences, and we establish the plausibility of hypotheses through a comparison with experience.

Now, for their solution, certain problems require hypotheses that cannot be deduced from the existing mathematics. Such hypotheses change mathematics in a profound and far-reaching way. For example, the problems of the infinitesimal calculus required hypotheses that could not be deduced from the (then) existing mathematics. They changed mathematics in a profound and far-reaching way. Hypotheses that cannot be deduced from the existing mathematics give rise to revolutions in mathematics. For more on revolutions in mathematics, I refer the interested reader to Cellucci 2022, chap. 10.

(3) Mancosu says: "The predominance" of mainstream "approaches to the philosophy of mathematics in the last twenty years proves" that "the 'maverick tradition'," namely the tradition originating from Lakatos, "did not manage to bring about a major reorientation of the field" (Mancosu 2008, 5). Indeed, "logically trained" mainstream philosophers of mathematics "felt that the 'mavericks' were throwing away the baby with the bathwater" (ibid., 5–6).

But this objection is invalid. The 'baby' thrown away is the assumption that mathematical reasoning consists of deductive reasoning. This assumption is untenable because it is refuted by Gödel's first incompleteness theorem. Moreover, the predominance of mainstream approaches to the philosophy of mathematics in the last twenty years does not prove that the maverick camp has failed. As Dewey says, "old ideas give way slowly" because they are "deeply engrained," but eventually they are abandoned because of "their decreasing vitality" (Dewey 1910, 19). Now, there are clear signs that the ideas of mainstream philosophy of mathematics have shown a decreasing vitality.

§ 9. — Heuristic Philosophy of Mathematics and Open Systems.

From the description of the main characteristics of heuristic philosophy of mathematics given above, it is clear that the basic assumption of heuristic philosophy of mathematics is that mathematics is problem-solving by the analytic method.

The basic assumption implies that mathematical theories are open systems. An open system is a system whose development involves receiving inputs from and delivering outputs to the outside. Thus, its development does not remain internal to the system, so the system is not a self-sufficient totality.

The basic assumption implies that mathematical theories are open systems because a mathematical theory initially consists only of problems to be solved and possibly other data already available. Its development consists in obtaining more and more hypotheses to solve the problems by non-deductive rules, and in checking that the hypotheses are plausible.

Since the hypotheses are obtained by non-deductive rules and non-deductive rules are ampliative, namely the conclusion contains something essentially new with respect to the premisses, the hypotheses are not implicitly contained in the problems and the other data already available. Moreover, the hypotheses need not belong to the same part of mathematics as the problems. They may belong to other parts of mathematics. So, the development of a mathematical theory may involve interactions with other mathematical theories.

§ 10. — Heuristic Philosophy of Mathematics and Gödel's Incompleteness Theorems.

The basic assumption, that mathematics is problem-solving by the analytic method, is unaffected and even confirmed by Gödel's incompleteness theorems.

The basic assumption is unaffected and even confirmed by Gödel's first incompleteness theorem, because the analytic method does not confine mathematics within the closed space of an axiomatic system. It lets mathematics develop in an open space, making use of interactions with other systems of knowledge. Indeed, in the analytic method, the solution to a problem is obtained from the problem, and possibly other data already available, by means of hypotheses not necessarily belonging to the same part of mathematics as the problem. Now, by Gödel's first incompleteness theorem, for any consistent, sufficiently strong, formal system, there are propositions of the system that are true but cannot be deduced from the axioms of the system. So, solving a problem of a given part of mathematics may require hypotheses from other parts.

The basic assumption is also unaffected and even confirmed by Gödel's second incompleteness theorem, because the analytic method does not assume that the solution to a problem is certain. Indeed, in the analytic method, the hypotheses for the solution to a problem are only plausible, therefore no solution to a problem can be certain. Now, by Gödel's second incompleteness theorem, for any consistent, sufficiently strong, formal system, it is impossible to demonstrate, by absolutely reliable means, that the axioms of the system are consistent. So, since we cannot know whether the axioms are consistent, no solution to a problem can be certain.

§ 11. — Other Advantages of Heuristic Philosophy of Mathematics.

In addition to being unaffected and even confirmed by Gödel's incompleteness theorems, heuristic philosophy of mathematics does not have the other shortcomings of mainstream philosophy of mathematics.

(1) Heuristic philosophy of mathematics accounts for the fact that solving a problem of a given part of mathematics may require hypotheses from other parts of mathematics.

For, according to the analytic method, the hypotheses to solve a problem need not belong to the same part of mathematics as the problem, they may belong to other parts of mathematics.

(2) Heuristic philosophy of mathematics accounts for the fact that a demonstration may yield something new.

For, according to the analytic method, the hypotheses for the solution to a problem are obtained from the problem, and possibly other data, by some non-deductive rule. So, they contain something essentially new with respect to them, because non-deductive rules are ampliative.

(3) Heuristic philosophy of mathematics accounts for the fact that new solutions, even hundreds of them, are often sought for problems for which a solution is already known.

For, according to the analytic method, a mathematical problem can be seen from different perspectives, each of which may suggest different hypotheses that may lead to different solutions to the problem. Each solution establishes new relations between the problem and other parts of mathematics, showing the problem in a new light. Therefore, a new solution to a problem is a contribution to mathematics.

§ 12. — Difference from the Philosophy of Mathematical Practice.

Heuristic philosophy of mathematics must not be confused with the philosophy of mathematical practice.

In recent years, the philosophy of mathematical practice has turned into a variety of different approaches, sometimes even in contrast with each other. Attempts to classify them have been made (see Van Bendegem 2014, Carter 2019), but they do not reveal a unity of the different approaches. A unitarian approach results only from the papers in Mancosu (2008), the first collection of essays on the philosophy of mathematical practice. Therefore, in what follows, by 'the philosophy of mathematical practice' I will exclusively mean that approach of the papers in Mancosu (2008), as presented in the Editor's Introduction.

Mancosu says that the philosophers of mathematical practice "do not engage in polemic with the foundationalist tradition" (Mancosu 2008, 18). In particular, they reject "the polemic against the ambitions of mathematical logic as a canon for philosophy of mathematics," and do not consider mathematical logic to be "ineffective in dealing with the questions concerning the dynamics of mathematical discovery" (ibid., 4).

The philosophers of mathematical practice are only "calling for an extension" of the foundationalist tradition to "topics that the foundationalist tradition has ignored," namely topics concerning "aspects of mathematical practice" (ibid., 18).

That does not mean that the three big foundationalist schools were "removed from such concerns" (ibid., 6–7). Thus, Frege's development of a formal language aimed at capturing all valid forms of reasoning occurring in mathematics "required a keen understanding of the reasoning patterns to be found in mathematical practice" (ibid., 7). Hilbert's "distinction between real and ideal elements" also "originates in mathematical practice" (ibid.). Brouwer's intuitionism originated "from the distinction between constructive vs. non-constructive procedures" which was prominent "in algebraic number theory in the late 19th century" (ibid.). The direct and indirect descendants of the three big foundationalist schools "are also, to various extents, concerned with certain aspects of mathematical practice" (ibid.).

However, the three big foundationalist schools and their direct or indirect descendants "were limited to a central, but ultimately narrow, aspect of the variety of activities in which mathematicians engage" (ibid.). Conversely, the philosophers of mathematical practice "cover a broad spectrum of case studies arising from mathematical practice" (ibid.,18). They extend the investigation "to a variety of areas that have been, by and large, ignored" but "are absolutely vital to an understanding of mathematics" (ibid.).

This, however, is a difference in quantity, not in quality between the philosophy of mathematical practice and mainstream philosophy of mathematics, therefore the former is continuous with the latter. As a result, the philosophy of mathematical practice inherits all the shortcomings of mainstream philosophy of mathematics.

§ 13. — Different Answers to Basic Questions.

As I have already said, according to heuristic philosophy of mathematics, the task of the philosophy of mathematics is primarily to give an answer to the question: How is mathematics made? And, subordinately to it, to the questions: What is the nature of mathematical objects, demonstrations, definitions, diagrams, notations, explanations, beauty, applicability, and knowledge?

Heuristic philosophy of mathematics and mainstream philosophy of mathematics give different answers to these questions. It would require much space to consider all of them, so I will only consider two of them, the nature of mathematical objects and the nature of mathematical definitions, which are strictly related.

§ 14. — Mathematical Objects.

First, I consider the nature of mathematical objects.

Mainstream philosophy of mathematics introduces and justifies mathematical objects on metaphysical grounds.

For example, Gödel introduces and justifies sets by saying that they belong to "a non-sensual reality, which exists independently both of the acts and of the dispositions of the human mind" (Gödel 1986–2002, III, 323). Yet, "despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory" (ibid., II, 268). Through it, sets "are known precisely, and general laws" about them "can be recognized with certainty" (ibid., III, 312, footnote 18).

On the contrary, heuristic philosophy of mathematics does not introduce and justify mathematical objects on metaphysical grounds. According to it, mathematical objects are hypotheses mathematicians introduce to solve mathematical problems by the analytic method. Like all hypotheses in the analytic method, these hypotheses must be plausible.

When mathematical objects, introduced as hypotheses to solve a problem by the analytic method, turn out to be useful also to solve other problems, they consolidate and acquire a stability that makes them independent of the problem for which they were originally introduced, and become subjects of study themselves.

Therefore, according to heuristic philosophy of mathematics, mathematicians do not accept or reject mathematical objects on metaphysical grounds, but because they are, or are not, functional to the advancement of mathematics.

For example, Bombelli introduced imaginary numbers as a means to solve cubic equations. At first, imaginary numbers encountered strong resistance. Like Musil's young Törless, several mathematicians objected that "the square of every number, whether it's positive or negative, produces a positive quantity. So there can't be any real number that could be the square root of a minus quantity" (Musil 1986, 76). Eventually, however, mathematicians accepted imaginary numbers, not on metaphysical grounds, but because they were functional to the advancement of mathematics. Thus, Gauss said "If these imaginary quantities were to be neglected", the infinitesimal calculus "would lose immensely in beauty and roundness", and we "would be forced to add very hampering restrictions to truths which otherwise would hold generally" (Gauss 1880, 156).

§ 15. — The Stipulative View of Mathematical Definition.

Now I consider the nature of mathematical definitions.

According to mainstream philosophy of mathematics, a definition merely stipulates the meaning of a term, it is only an abbreviation, so it is always correct, and can be eliminated.

This view of definition, which is known as 'the stipulative view of definition', goes back to ancient Greece. Thus, Aristotle says: "One type of definition will be the statement of what a name" means, "for example, what triangle means" (Aristotle, *Analytica Posteriora*, B 10, 93 b 30–32).

But Aristotle considers this kind of definition only to criticize it. Indeed, he argues that, if a definition were simply a statement of what a name means, this would have the following undesirable consequences.

(1) "There would be definitions of non-substances and of things which do not exist, because it is possible to mean also things which do not exist" (Aristotle, *Analytica Posteriora*, B 7, 92 b 27–30).

(2) "All expressions would be definitions. For, it is possible to assign a name to any expression whatever, so all our discourses would be definitions," for example, the entire poem "*Iliad* would be a definition" (ibid., B 7, 92 b 30–32).

(3) "No demonstration could demonstrate that this name makes known this thing; therefore, definitions could not make known this thing either" (ibid., B 7, 92 b 32–34).

To the stipulative view of definition, Aristotle opposes the essentialist view of definition, which is based on Aristotle's fundamental assumption that "to know a thing is to know its essence," since "any single thing and its pure essence coincide" (ibid., Z 6, 1031 b 19–21). So, the object of science is to know the essence of things. Then, in particular, "a definition is said to be a statement of the essence of a thing" (ibid., B 10, 93 b 29).

Then, Galileo replaced Aristotle's fundamental assumption that the object of science is to know the essence of things by the fundamental assumption that gave rise to modern science: the object of science is to know phenomenal properties of things, mathematical in kind. Therefore, "the definitions of mathematicians" do not state the essence of things, they are only "an imposition of names, or we might say abbreviations of speech" (Galilei 1968, VIII, 74). So, they are arbitrary stipulations and, "being arbitrary, can never be bad" (ibid., IV, 700). Thus, Galileo revived the stipulative view of definition.

The stipulative view has prevailed in the past century especially as a result of Hilbert's replacement of the material axiomatic method with the formal axiomatic method.

Indeed, according to Hilbert, a "theory has nothing to do with the real objects and with the intuitive content of knowledge; it is a pure thought construct, of which one cannot say that is true or false" (Hilbert 2013, 435). So, "the axioms can be taken quite arbitrarily" (Hilbert 2004, 563). And a definition is "a mere explanation of signs" (Hilbert and Bernays 1968–1970, I, 292). Thus, definitions are arbitrary stipulations.

§ 16. — Shortcomings of the Stipulative View of Mathematical Definition.

Despite its prevailing in the past century, the stipulative view of mathematical definition has important shortcomings.

(1) The stipulative view does not account for the fact that finding an adequate definition can make the difference in discovering a solution to a problem.

For example, let us consider the definition of a sphere. There are two historically important definitions of a sphere, one by Plato, Aristotle, and Theodosios, the other by Euclid.

Plato, Aristotle, and Theodosios define a sphere as a solid figure with every point on its surface equidistant from its centre.

For example, Theodosios defines a sphere as "a solid figure contained by one surface, such that all the straight lines falling upon it from one point among those lying inside the figure are equal to each other" (Theodosios, *Sphaerica*, I, Def. 1, ed. Heiberg).

On the contrary, Euclid defines a sphere as a solid figure generated by a semicircle revolving about its diameter.

Indeed, Euclid defines a sphere as "the figure comprehended when, the diameter of a semicircle remaining fixed, the semicircle is carried around and restored again to the same position from which it began to be moved" (Euclid, *Elementa*, XI, Definition 14).

On the other hand, while Euclid defines a sphere by referring to motion, he defines a circle without referring to motion.

Indeed, Euclid defines a circle as "a plane figure contained by one line, such that all the straight lines falling upon it from one point among those lying inside the figure are equal to each other" (Euclid, *Elementa*, I, Definition 15).

Why does Euclid define a sphere and a circle in different ways? This is because Euclid's definition of a sphere plays an important heuristic role in Euclid's solution to the problem about the five Platonic figures.

Finding a solution to that problem was very important to Euclid because, as Thomas says, the solution to the problem about the five Platonic figures is "the culmination of the work" of Euclid's *Elements* "as a whole" (Thomas 2014, 234).

By referring to motion, Euclid's definition of a sphere guided him to find a solution. Thus, such definition made the difference in discovering a solution to the problem. This explains why Euclid defines a sphere and a circle in different ways.

This cannot be accounted for, as the stipulative view claims, a definition is only an abbreviation.

(2) The stipulative view does not account for the fact that mathematicians often use concepts for a long time, even centuries, before they can find a suitable definition for them.

For example, as Grabiner says, "Fermat implicitly used" the derivative; "Newton and Leibniz discovered it; Taylor, Euler, Maclaurin developed it; Lagrange named and characterized it; and only at the end of this long period of development did Cauchy and Weierstrass define it" (Grabiner 2010, 159–160). So, "the historical order of development of the derivative is the reverse of the usual order of textbook exposition," in which "one starts with a definition, then explores some results," thus "a definition is often the end, rather than the beginning, of a subject" (ibid., 160).

This cannot be accounted for if, as the stipulative view claims, a definition is only an abbreviation.

(3) The stipulative view does not account for the fact that mathematicians often give definitions that afterwards turn out to be incorrect.

For example, Jordan defines a curve in a way that corresponds to the idea that a curve is what is generated if a point runs along in continuous motion. The motion of the point will be completely described by stating how the two coordinates x and y of the point depend on time t.

Indeed, Jordan defines a curve as "the succession of points represented by the equations $x = f(t)$, $y = \phi(t)$, where f, ϕ are functions of an independent variable t. If these functions are continuous, the curve will be called continuous" (Jordan 1887, 587).

However, Peano defines something that is a continuous curve according to Jordan's definition, but "goes through every point of a square" (Peano 1973, 144).

Now, a curve is a one-dimensional object, while a square is a two-dimensional object. Since, on Jordan's definition, there is a curve that is a square, then Jordan's definition of curve or continuous curve is incorrect.

This cannot be accounted for if, as the stipulative view claims, a definition is always correct.

§ 17. — The Heuristic View of Mathematical Definition.

An alternative view of mathematical definition is the heuristic view. It derives from the heuristic view of mathematical objects.

Indeed, if mathematical objects are hypotheses human beings make to solve mathematical problems by the analytic method, then so are the mathematical definitions that introduce them.

Thus, mathematical definitions are hypotheses that are made to solve mathematical problems by the analytic method. Like all hypotheses in the analytic method, such hypotheses must be plausible.

§ 18. — Advantages of the Heuristic View of Mathematical Definition.

The heuristic view does not have the shortcomings of the stipulative view.

(1) The heuristic view accounts for the fact that finding a suitable mathematical definition can make the difference in discovering a solution to a problem.

For, a mathematical definition is a hypothesis, and finding a suitable hypothesis is a crucial step towards finding a solution to a problem.

(2) The heuristic view accounts for the fact that mathematicians often use concepts for a long time, even centuries, before they can find a suitable definition for them.

For, finding an adequate definition can make the difference in discovering a solution to a problem. So, finding an adequate definition may require as much effort and time as finding any hypothesis capable of yielding a solution to a problem.

(3) The heuristic view accounts for the fact that mathematicians often give definitions that afterwards turn out to be incorrect.

For, a mathematical definition is a hypothesis that is plausible, and a hypothesis that is plausible at one stage may become implausible at a later stage, when the arguments against the hypothesis become stronger than those for it.

Thus, mathematicians thought that Jordan's definition of a curve was plausible, but then Peano gave an argument that made it implausible.

§ 19. — Conclusion.

From what I have said, it seems fair to conclude that heuristic philosophy of mathematics is more adequate than mainstream philosophy of mathematics.

I have argued for this by considering the nature of mathematical objects and definitions. But I could have argued for it equally well by considering the nature of mathematical demonstrations, diagrams, notations, explanations, beauty, applicability, and knowledge. For them, I refer the interested reader to Cellucci 2022, chap. 10–17.

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