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Ways of thinking in mathematics

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Abstract. All mathematicians, whether they are engaged in teaching, in researching or in both at the same time, must master specific ways of thinking, of reasoning. These are ways of thinking that can be divided into three categories: reasoning and problem-solving; organizing already acquired knowledge; and characterizing, constructing or defining new elements and concepts. These are the modes that are presented in this essay, with a very brief indication of the historical moment in which they are constructed. The obligatory mastery of these modes of thinking, by itself, does not lead to mathematical creation; the imaginative and creative power of the mathematician is involved in this. Finally, we point out the appearance of an ontological problematic from the inversion that took place in the 19th century: are mathematical entities discovered or constructed? It is a problematic that has become a theme throughout the twentieth century about mathematical doing.

Man

Western man, from birth to death, lives in a space built according to Euclidean metric geometry;
Man, from birth to death, lives in an arithmetical world of recurrences, counts, weights and measures;
Man, from birth to death, lives in a topological space of open and closed neighborhoods, of insides and outsides, of frontiers;
Man, from birth to death, lives in a space of structures, of groups and their relationships;
Man, from birth to death, lives in a world of statistics, of percentages, of sampling;
Man, from birth to death, is one more element of the big-data that is built by companies and governments;
Man, from birth to death...
....
Western man, from birth to death, lives in a mathematical world.

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In the last third of the twentieth century there was a rupture in the way of thinking about mathematics. From the study of its foundations according to the formalist, intuitionist and logicist schools, mathematics came to be approached as a product of the mathematician, as a work or task within certain social frameworks. In 1972, at the zenith of a Bourbakist formalism, I devoted a paper to what I called mathematical styles. Since then, and so far in the 21st century, it has become a cliché to speak of styles, of the practice of mathematics.

From these positions a question arises: if it is stated that mathematics requires that someone practices it and, in this practice, he or she obtains a product, the mathematical content, is there a specific way of thinking, of making, in order to obtain this product? If so, what is it? I start from the conviction that every scientific discipline has its own way of making, of working, a characteristic way that must be mastered and managed by those who dedicate themselves to the cultivation of that discipline.

The aim of this essay is to briefly outline the ways of mathematics doing and its origins. With the precision that every mathematician, besides mastering these ways of thinking, has to study and know the content of his discipline with which he or she obtains what is considered to be mathematical experience. It is clear that the creative mathematician, from this experience and in order to construct new and beautiful mathematics, must possess a great creative imagination to pose new problems, questions and approaches, establish analogies between different fields and even provoke authentic epistemological ruptures.

This is what Abel did when he turned the question of quintics upside down, thus giving rise to the future theory of groups; or Dedekind, when he tried to construct mathematical analysis without resorting to geometry, giving rise on the one hand to the theory of sets with Cantor and on the other to the arithmetization of analysis in line with Weiertrass and Kronecker; or Grothendieck when, instead of defining an object and exploring an internal structure of that object, goes on to define the category of all interacting objects and studies the internal structure of that category. It is an epistemological inversion that implies moving from the object in itself to the multiple object through an associated representable functor. With this inversion Grothendieck constructs algebraic geometry in the categorical manner, as Fernando Zalamea has emphasized in his wonderful works.

§ 1.

Plato, in the second mathematical passage of his dialogue *Menon*, takes for granted that geometricians reason in a specific way: by hypothesis. It is what today, with nuances slightly different from the original Greek language, we qualify as assumptions or postulates. It is the starting point of reasoning supported, implicitly, in the principles of contradiction and excluded middle, namely in the principles $\neg(A \wedge \neg A)$ and $A \vee \neg A$.

The geometrician establishes some assumptions or postulates of which he or she does not know beforehand whether they are true or not; from them, he or she reasons deductively and, having obtained a conclusion, whoever accepts the assumptions or postulates has to accept that conclusion. If he or she reaches an incorrect conclusion, he or she discards the hypothesis or hypotheses from which he or she started, adopts new ones and repeats the deductive, derivative process. This way of working can be described as trial and error.

What Plato has exposed is one of the key modes for the work of every mathematician: he imposes the condition that every mathematician must have a great capacity to raise hypotheses, assumptions, conjectures... from which to start and then maintain a rigorous derivative process.

There are several points in the above: first of all, accepting that the hypothetico-deductive mode of reasoning is that of the geometrician, of the mathematician. Moreover, if we associate the principle of contradiction to this mode of reasoning, we have one of the most powerful methods of mathematical reasoning: the reduction to the absurd; if one wants to prove A, one adopts $\neg A$ as a hypothesis; one derives with all possible rigor and arrives at a contradiction; consequently, one must accept its negation, that is to say, $\neg\neg A = A$. This is what Newton qualifies in *Principia mathematica* as "the tediousness of deducing perplexed demonstrations ad absurdum, according to the method of the ancient geometers". (Scholium to Lemma XI, Book I, section I)[(1)]. The method of the ancients is the other side of the hypothetico-deductive method: from the approach of problems, conjectures to be solved and propositions to be demonstrated, we move on to the attempt of solution or demonstration, using one of the best instruments that the mathematician has, that of reduction to the absurd.

It seems that by this method the Pythagoreans demonstrated the incommensurability of the diagonal of the pentagon with respect to the side, or the diagonal of the square with respect to its side. As an immediate problem, they must prove that $\sqrt{2}$, for example, is a number; making use of the reduction to the absurd, and by successive hypotheses, it is proved that if it is odd one arrives at contradiction; the same holds for the hypotheses of even or rational number; consequently, it is not a number and, hence, we deduce the incommensurability that goes against one of the essential Pythagorean principles; but reason imposes itself: $\sqrt{2}$ is not number, consequently there is no commensurability. However, with the diagonal of the square one can obtain a geometrical property with an accuracy that is impossible from the pure numerical: to obtain a square of an area that is double of a given one. This is what the first geometrical passage of Plato's *Menon* will show. New consequence: the Pythagoreans establish that it is possible to reason with two different approaches: in a pure geometrical way and in a pure arithmetical way, with two methods that much later Aristotle will qualify as two demonstrative genres: geometrical, arithmetical. As two different genres they are maintained until about the 16th century.

[(1)]Isaac Newton, Newton's Principia : *The mathematical principles of natural philosophy,* New York, Daniel Adee, 1846, p. 102.

What I am interested in emphasizing is that in Greece conceptual mathematics arises related to two ways of thinking: the direct hypothetico-deductive and the reductio ad absurdum. And so, with two distinct genres: geometric and arithmetic, and a caveat: what reason imposes is accepted even if what is obtained goes against admitted principles. On the other hand, it is a way of reasoning that requires the figural diagram: one must start from the diagram, knowing that the figure — the triangle that is drawn and erased — is nothing more than a representation, a scheme, but it is indispensable for this way of reasoning.

Euclid, in the third century, turns this hypothetical-deductive method from a process or method of reasoning into an organizational process. Euclid starts from mathematical knowledge already acquired and organizes part of it for teaching in his book *Elements.* He chooses definitions, common notions and postulates for the geometrical part and gives definitions for the remaining parts of the work. Together with the diagram, he exposes, deductively, this knowledge. It is an organizing principle that in 1972 I described as geometric style. These *Elements* do not include all the mathematical knowledge known at the time, nor do they establish postulates for arithmetical reasoning.

It is an organizational approach that raises numerous problems such as, for example, whether the postulates chosen are sufficient to account for the sectoral knowledge adopted, whether these postulates are independent... On the other hand, there was a turning point started by Aristotle: scientific disciplines must be truthful and, therefore, they cannot start from postulates but from axioms, from self-evident truths. The hypothetical-deductive method becomes the axiomatic method. With this, those who follow the Aristotelian line accept an inflection: axioms cannot be denied because it would be foolish to deny the evidences. It is a line that became predominant and was even adopted by Kant.

The *Elements*, the work that Euclid composed for teaching in the third century, has been the most influential book in the Western world. Considered an example of the rigor of reasoning, of expository rigor, it has been used in many other disciplines and even in other non-scientific fields, and of course in teaching.

§ **2.**

A time jump and we reach the sixteenth and seventeenth centuries. Exceptional moments for the mathematical and scientific work that is being developed in these times. We must focus on René Descartes. Descartes creates another way of thinking and making mathematics, a way that in 1972 I called algebraic-Cartesian style. He breaks and at the same time unifies the two demonstrative genres established since the Greek world.

Without axioms, operational rules are established for the use and handling of the symbols of the alphabet, together with those of the decimal number system, in such a way that they represent geometric objects both in the plane and in space. An expression such as $3x + 4y + 5 = 0$ is that of a line in the plane, a line whose general expression is given by $ax + by + c = 0$. Here, x, y represent variables; a, b, c the associated parameters. Two expressions of this type are those of two straight lines whose relative positions will be studied according to the parameters, for the handling of which a new object, the determinant, will be constructed.

If the use of figurative diagrams — the triangle, the circumference... — were essential in the geometric style, now it is even more essential to use the first letters of the alphabet to represent parameters and the last letters for variables. It is necessary to reason on the basis of these symbolic diagrams.

An expression such as $ax + by + c = 0$, in the plane, represents a straight line. There is something more: to make that graphical representation it is written as $y = -\frac{a}{b}x - \frac{c}{b}$ and it is held that to each value of x corresponds a value of y; so the points $P(x, y)$ will be obtained in any given reference frame in the plane. The last expression goes beyond this geometric role because what it represents is a new concept, that of function, $y = f(x)$, with x as independent variable and y as dependent, in this case.

The function is a concept, an instrument conceptual that conditions a new way of thinking in both mathematical and scientific doing. This is what Newton will express in his *Principia*, where he shows, with the formulation of his laws as functions, that it is a question of mathematical doing. It is a scientific endeavor that historically has been approached as purely quantifying because it establishes as a dogma the division between subjective qualities — smell, taste, color, texture... — and objective magnitudes which are quantifiable and measurable — force, acceleration, quantity of matter or mass...

Certainly it is a quantifiable activity, but its power goes beyond quantification because it is a functional activity. What matters are not the isolated concepts but the concepts in a network, in mathematical relations. Thus, the second law is established as the functional relation between force, mass, acceleration; what we have is a differential equation as an expression of a law of physis, of nature.

Thus, together with a way of thinking, of an algebraic making, we have the creation of another way of thinking, now a functional one. The trajectory of a mobile in the space will be nothing but the representation of a function, and that trajectory, that function, can be represented in an Euclidean metric geometric space with a given referential system. We have then two ways of thinking that are so linked to each other that sometimes we use the algebraic in the functional and vice versa.

The functional mode of thinking, on the other hand, is initially elaborated on the basis of another geometrical image: that of indivisibles. The surface that determines a curve in the plane is considered to be constituted by infinitesimal rectangles, increasingly of smaller area for a greater approximation to the given surface. With this idea, and that of rotation with respect to the abscissa axis, Blaise Pascal will outline the definite integral calculus, giving even the rules of integration by parts and by change of variable.

Immediately, Leibniz makes an epistemological inversion by studying Pascal's writings and goes on to construct the differential calculus as the inverse of the integral. He does so with a writing that breaks with the style of indivisibles used by Pascal and Newton. Leibniz imposes the writing that we follow today for the infinitesimal analysis and that will have in the work of L'Hôpital the first reference book.

Handling the style of indivisibles, but in the language of fluxions for differential calculus, Newton, from his laws and definitions, will approach the motion of bodies in Euclidean geometric space as represented by functions, and for its study he will handle the serial development of these functions with a given remainder.

The sixteenth and seventeenth centuries were exceptional times. In addition to the ways of thinking coming from Greece, Blaise Pascal produces another specifically mathematical mechanism. In his *Treatises on Arithmetic*, in order to demonstrate properties of the sums of the series appearing in his arithmetical triangle, Pascal will handle, for the first time in history, the method of complete induction. This principle of complete induction is, for some later mathematician, the authentic way of thinking, the most specific, of making mathematics.

There is one more element: in an epistolary exchange, Pascal and Fermat give way to the Calculus of Probabilities. Immediately, the *Ars conjectandi* is published, the use of statistics appears... Another way of approaching knowledge emerged.

From now on, the algebraic and functional modes of reasoning, the method of complete induction, the principles of the calculus of probabilities and statistics are added to the hypothetico-deductive method and to the method of reductio ad absurdum. In his or her starting point, in the search for assumptions or hypotheses, in the search for conjectures, the mathematician must now add the search for relations and functions between magnitudes that can be quantifiable or have a high degree of probability.

In this task, the mathematician has two conditioning factors: he or she must start from the diagram, whether figurative or symbolic, in order to carry out his or her work. In addition, and from his or her starting point, he or she will have a singular, concrete object, such as studying the behavior of a specific function, solving an algebraic or differential equation, trying to solve one of the millennium problems, studying a Fourier transform... It is a way of thinking or making, of working, that I have described elsewhere as figurative making.

I must observe that the mathematician, from his or her very starting point, no longer has a more or less delimited or unique field: from these moments his or her work will split into different fields. This division will become, in some moments, deeper in order to give way to different styles, to different ways of making mathematics.

§ 3.

The previous way of making mathematics was radically criticized during the 19th century. Richard Dedekind observes that in his lectures on differential and integral calculus, he resorts to graphical representation, an instrument that proves to be indispensable. Dedekind wonders if, in fact, this representation is so essential. His answer, as we all know, is that we must dispense with this representation: we must go to an expression proper to Analysis.

In this answer Dedekind provokes a profound epistemological rupture, which Weiertrass and Kronecker will also make on their side: the elimination of the geometric image leads to a process that has been called the arithmetization of analysis, but which is deeper than the simple process. In his work to characterize real numbers Dedekind introduces the cuts, the classes or sets of rational numbers. A process that, together with others coming also from the study of functions, leads Dedekind, Paul du Bois-Reymond... and, above all, Cantor to elaborate the theory of sets.

In these works, a new way of thinking is produced, which is no longer figurative, but global. Now it is necessary to start from sets, classes, extensions of concepts... and not from concrete objects as in previous ways of thinking. It is a global way of thinking in which objects become structures built on sets or classes. For some, it implies moving from using real numbers to thinking about structures.

On the other hand, another turning point takes place in terms of the mathematical content itself: by trying to clarify and define the concepts of Analysis without calling upon the geometric figurative image, a question of foundation arises. Mathematicians move on from attempting to found the Analysis to the attempt to found all mathematical work with a break with respect to the previous work; we no longer have recourse to the geometric figurative diagram, although we do have recourse to the symbolic diagram. In an intrinsic work of his own, some mathematicians stop at the natural number, like Kronecker, but others go further and look for a foundation for the natural numbers themselves. Frege thought he found that foundation in logic and in the extension of concepts, which is nothing other than logic and set theory. A new formal or mathematical logic that ends up becoming a new object of work for the mathematician. For his part, Cantor and his followers will try to find this foundation in set theory.

Both attempts end up in paradoxes or associated difficulties but, in any case, the question of foundations, in these first moments, is the subject and work of mathematicians.

But the mathematician, besides discussing and elaborating these processes of arithmetization, formal logic, set theory, and organizing this knowledge in logical or conjunctive axiomatized theories, is also concerned with the construction of new fields of work – topology, algebra... — in which different approaches, ways of doing things and styles will be manifested. All these fields are supported by a new way of making: a global way of making where what matters is not the set itself, but the set with a given structure together with the morphisms, the links between these structures. It is a way of making that also requires a new ideographic language: it is necessary to specify, with clarity, the membership of an element to a set, the existential and universal quantifiers, the conjunction and disjunction operators, the initial and final sets...

§ 4.

The ways of structuring a set imply new ways of characterizing, of defining objects, because in the epistemological inversion produced, not only new ways of reasoning are created, but also new ways of characterizing what is to be handled. The traditional modes of definition are the *nominal* or stipulative, which gives a name to what exists or is constructed; the *descriptive* or dictionary definition, which, in general, is never complete and can even be circular; and the definition considered as the authentic definition, which is the *essential* one: the one that establishes both the genus and the specific difference of the object defined by what, from the traditional logic, is intended to characterize the essence of that object. These are the definitions accepted in traditional logic, the one canonized since Aristotle and scholasticism.

But I insist that we no longer start from the singular entity, for which these types of definition are valid; now we start from sets or classes, and this forces us to construct new ways of defining or constructing the new mathematical entities. Thus, together with these characterizations, the following will be established in the new mathematical practice:

By *abstraction:* Given a set or class A, a relation $\mathscr{R}$ of equivalence is defined between its elements; this relation causes a partition of the set in equivalence classes $A/\mathscr{R}$ where each class is considered a new object. It is a mechanism by which Gauss had already established his congruence classes. It is the process that, for example, will be used to construct, from the set theory as a basis, the natural numbers, from the natural numbers, the integers, from these, the rational numbers and so on...

Definition by abstraction is a mechanism that involves the formation of new objects and structures, of new concepts. A mechanism that, from a global point of view, is linked to the notion of function: to every function is associated not only an equation, but also an equivalence relation in the domain of that function: $x\mathscr{R}y$ if and only if $f(x) = f(y)$.

On the other hand, it could be considered as the proper mechanism for the elaboration of classificatory concepts and, thus, as a formal justification for the process considered as abstraction. Of course, it poses a problem for some realist philosopher: these objects are not singular entities but sets, classes; even more, classes of classes of classes...

By *recursion:* it is a key mechanism as an element of the definition for arithmetic where the operations of addition and multiplication are defined by recursion and their properties are proved by complete induction. I would refer to the work of Peano in his formalization of arithmetic, to Poincaré's work on mathematical reasoning...

By *postulates, implicit or axiomatic definition*: together with the organizational character established by Euclid for geometry, the hypothetico-deductive method becomes, from the 19th century onwards, and after the discussion of the role of axioms in the different geometries, a constructive method of concepts. The discussion of axioms in their role of conventions or disguised definitions makes them end up being approached as constitutive and, at the same time, regulative elements of some fields of play. Thus, to characterize a structure as that of a group is to give a set and, in it, an operation that satisfies certain conditions, postulates or axioms; from them, so to speak, begins the game of deriving properties in that field of play. The group structure has been characterized by means of an axiomatic definition; the equivalence relation that is imposed in the definition by abstraction is characterized according to some conditions or postulates, to an implicit definition.

It is a method of construction of structures, of concepts that has become one of the most important tools for mathematical work from the global approach. It is an approach that has been constituted as such since the appearance of non-Euclidean geometries in the 19th century and that has led to the discussion of the role of axioms. A discussion in which, depending on which postulates one defends, one will have affine, metric, projective geometries... This is what Hilbert will clearly show in his *Fundamentals of Geometry* with a beginning such as "Let us think of three different systems of entities..." to later establish the axioms corresponding to each of the previously established groups.

It is a line of work that moves to more algebraic fields from which these geometries will be characterized as groups of transformations, in such a way that it can be affirmed that a geometry is nothing other than a certain group of transformations. It is a process in which the algebraic approach to geometry is radicalized.

The hypothetico-deductive method thus acquires a new dimension: it is reasoning, it is an organizational tool of already acquired knowledge, but it also becomes an instrument for the characterization of new entities, of new mathematical objects and concepts that are already geometries, groups of transformations...

§ 5.

The modes of thinking described above, structured in three sections, that of reasoning, demonstrating and solving problems, that of organizing already established fields of knowledge, and that of constructing and defining concepts, are modes of thinking proper to the mathematician. They must be handled by those who are dedicated to teaching, to educating citizens of whom some, very few, will be inclined to mathematics in the future; they must be assimilated by the creative mathematician who, in addition to learning the mathematics of his or her environment, of his or her time and circumstances, in addition to effort and work, must have a very strong creative imagination and, of course, luck in his work.

With a nuance: both the mathematician who focuses on teaching and the one who, in addition to teaching, engages in research, do so with different sensibilities: the split produced from the origin by the Pythagoreans between geometry and arithmetic reflects a very deep separation. It does not apply only to the modes of demonstration — or genres in Aristotelian terms —. There are mathematicians with a special sensitivity for the continuum — Riemann would be an example —; others for the algebraic — some members of Bourbaki would be models —; others for the arithmetical — I would cite Spanish number theorist Laureano Pérez-Cacho (1900-1957) —. There are different sensibilities, situated in social contexts that may also be different, to build, to materialize a common making for all, a common making that is mathematics.

§ 6.

The profound reversal that took place at the end of the 19th century and that led to globalization can be considered to be parallel to that of Western society. From a historical-sociological approach, it can be observed that throughout the 19th century, and as a consequence of techno-scientific advances, capitalist democracies developed in the Western world with their proposals for new nationalisms and states. This is the origin of political parties, trade unions, social classes... The individual loses prominence in the face of the structures of the new collectivities.

In this inversion, and in the mathematical field, in the global way of making things, the mathematicians' problem of establishing their own way of making things has arisen. A problem that did not exist until then, because formerly the starting point was the datum of the singular entity, whether arithmetic, geometric, analytical... or coming from fields such as physics, astronomy in the form of celestial mechanics... I have indicated that the classical definitions were valid for this approach. However, the inversion makes the mathematician have as a starting point the set or class and define in it an operation, a relation or some axioms to give it a certain structure. The datum of this set also supposes the adoption of the actual infinite as a given entity; moreover, different types and scales of infinite: the actual numerable infinite, the continuous, the transfinite cardinals and ordinals...

It is the process, as I have indicated, for the construction of the natural numbers; from the set of all the pairs of NxN and with a given equivalence relation, we have the integers that are shown as classes of classes of pairs of naturals; from the integers, and by the same procedure, we obtain the rationals... On his side, when Weiertrass characterizes a function with an infinity of points of discontinuity with impossibility of graphic representation and gives way to what with Hermite are considered teratological functions, this actual infinite is also handled. The immediate question arises: do these constructions make any sense, or are they what Goya wrote in one of his engravings "The sleep of reason produces monsters"?

The mathematician is confronted, in a profound way, with the existential, ontological problem of the entities he handles. In classical logic, definition has an Adamic role: Adam gave names to already existing beings. It is the mechanism that, for example, Linnaeus used in his botanical classification: genus-species now family-species... A basically descriptive mechanism of something that does not offer, in principle, difficulties because it is given beforehand. Now it seems that by means of definitions the mathematician constructs structures that are not given, that are not found in ordinary life, just as one does not find that actual infinite that one has at the starting point. The new entities are no longer, as Aristotle intended, and with him the later tradition, abstraction of objects of ordinary life, they are not the abstract ideas obtained by the creative mathematician from that ordinary life.

As early as 1882, Paul du Bois-Reymond constructs his work *General Theory of Functions* as a confrontation between two positions for the construction of analysis, which he calls idealist and empiricist. But it will be Poincaré, in the early years of the twentieth century and in his harsh criticism of Russell's and, above all, Cantor's positions, who will highlight the existential, ontological problem that the new way of doing implies.

For Poincaré, the Cantorians appear to be authentic realists because for them the mathematical entities, among which the actual infinite is necessarily found, have their own independent existence, in such a way that the "geometrician does not create them, he discovers them". Because they have this existence independent of the mathematician, these entities exist even if no mathematician exists. A realistic position that holds for the mathematical world as well as for the physical, material world. In the classical, traditional way, Poincaré will suggest that the world in which these entities exist is similar to that of the ideal forms of which Plato spoke. In this way, what will be called "mathematical Platonism" is introduced, a name that is not very fortunate in terms of its historical attribution.

To the Cantorian realist position Poincaré will oppose those he calls the Pragmatists. They are those who only admit objects and entities that can be defined by a finite number of words. For this reason, they consider that a (mathematical) object exists only when it is thought, and it would not be possible to conceive of an object thought independently of a subject that thinks it. Poincaré will affirm that this position is that of an idealist. As opposed to the Cantorians, with their existential realism, the pragmatist admits that the thinking subject is a human being — "or something resembling a human being", Poincaré will ironize — who is a finite being, and therefore "infinite can have no other meaning than the possibility of creating as many finite objects as one wishes".

In fact, Poincaré's criticisms go in parallel with the appearance of paradoxes both in set theory and in Frege's logicism, which give rise to various attempts to overcome them. What I am interested in emphasizing is that in Poincaré's critique two elements are raised that intermingle in these attempts to overcome the paradoxes, in the subsequent attempts to clarify the difficulties and in the mathematical work itself. On the one hand, there is an epistemological question, which is centered on determining which are the proper modes of mathematical thinking, which are the modes with which he carries out his work and with which he obtains a certain mathematical knowledge; it is the question that Poincaré would point out when he indicates that mathematical entities are those that can be obtained by definition by a finite number of words, those obtained by finite methods, since man is a finite being.

On the other hand, there is the purely existential, ontological question, which focuses on the type of existence of mathematical objects and entities, whether or not this existence is independent of the subject who thinks and handles them, whether or not they are independent of the mathematician; in the latter case, if this independence is admitted, the epistemological problem arises here: how this independent world is reached...

They are two intermingled questions because from the epistemological approach, by saying that mathematical entities are obtained through a certain process, one is affirming, at the same time, the constructive, existential role of the mathematician: one gets to know what one is constructing... With an added problem: among other things Poincaré will recall a very classical dictum: definition does not imply the existence of what is defined. And this will force to establish conditions to assure the existence that the definition implies. Conditions that go from obtaining the data of a model to trying to demonstrate consistency for the defined. These are issues that have been intermingled throughout the twentieth century in the different positions that have been established to overcome the initial paradoxes.

With Poincaré and in line with what I have maintained since 1972, I consider that mathematics involves the work of a thinking being, the mathematician. It is a work or making by which he or she obtains a product, mathematical knowledge, in a given historical context and circumstances. For his or her work he or she requires a previous mathematical experience achieved through an apprenticeship, although, with the French mathematician, "the mathematician is born, not made".

In his or her construction, the mathematician has to carry out a work, sometimes very hard work, to end up obtaining the creative idea. In this apprenticeship, with the experience that it entails, the mathematician learns to handle some regulating elements such as the ways of thinking presented here, or the actual infinite.

In any case, and from the sensitivities of each mathematician, he or she has to start from that mathematical experience. This experience is obligatory in order to be able to carry out his or her work, be it teaching, solving a problem, obtaining a new theorem or making an epistemological turn with the construction of a new approach and new contents... but always in order to make mathematics.

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