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$M \times \Phi$

Some Personal Remarks

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Philosophy is interested in the big question of human knowledge, its sources and limits; and, mathematics being a central ingredient in the elaboration of human knowledge, a philosopher can hardly avoid questions about mathematical concepts and methods (and objects too). The traditional debates surrounding this issue have been trapped between the extreme positions of platonism (math depicting, or trying to depict, a fully independent realm of objects and their interrelations) and mentalism (math as a human creation, its objects mere products of the mind). Yet I believe that none of these extremes is true to the topic: math originates in our experience, both passive ("outer") and active experience, which leads to discoveries about patterns in the world; but math grows through the assumption of hypothetical states of affairs, and it is devoted to exploring the properties of such hypothetical 'world pictures', developing methods that help in the exploration.(1) Thus mathematics is neither just discovered nor merely invented. The mathematical experience presents us with the phenomenon of discovering features of structures that we have partly constructed.

(1) See Feferman 2014, Ferreirós 2022.

§ 1. — Mathematics and nature.

About a century ago, there were attempts to establish that all human knowledge that is not empirical, can be reduced to logical laws and definitions (expressing the "conceptual analysis" of the meanings of crucial words), i.e. consists of analytic truths. But the underlying idea of a pure and transparent logico-linguistic rendition of empirical states of affairs (a kind of mirroring without distortion) is very likely to be unrealistic. Human knowledge seems to be more a matter of *synthesizing* models of the phenomena, models where we employ hypothetical assumptions (distortions, perhaps) in the effort to capture basic traits of the empirical or the given. This was, after all, the message that Poincaré tried to convey 120 years ago in *Science and Hypothesis* (Poincaré 1902): Synthesis, not mere analysis; conventions, not merely the empirically given and the a priori.

The distinction between pure and applied math is in large measure a product of recent institutional developments, not fully separable from some ideological assumptions. As 'purely' mathematical journals, departments, and congresses rose and grew from the 1830s onward, stably supported by our industrial societies, the ideology of pure math became dominant. But no area of human knowledge is fully separable from the rest, and it's impossible to make sense of mathematical knowledge around the year 2,000 without considering its history — a history that cannot be understood without the intricate links between math and knowledge of nature.(2) Thus, defending the honor of the human mind is fully compatible with partaking in the exploration of natural phenomena: Fourier and Jacobi are doomed to understand each other, in the end.

One could of course establish an artificial division of topics, regarding as "mathematical" only what has to do with logical deduction of consequences of established axioms, and considering non-mathematical, "scientific," what relates to the creation of models, the introduction of hypotheses eventually solidified in axioms. But imposing such a division onto real historical figures (Descartes, Newton, Gauss, Riemann), we would be cutting them in halves. By this criterion, Riemann was a 'scientist' most of the time, not a mathematician; Zermelo would have been a 'scientist' when he

(2) Again a message of Poincaré and Weyl (1951) — for examples, just consider the number systems (real or complex), the continuum, the function concept, or Fourier series.

realized the important role played by the Axiom of Choice, and a mathematician when he worked on his systematic presentation of an axiom system. The two moments can be distinguished, indeed, but the reality of mathematical (and scientific) practices is that both are interconnected.

§ 2. — The problems of growth and objectivity.

The growth of mathematical knowledge is for me one of the big questions, in many directions. One aspect is the problem of cognition and mathematics, which of course is well known in connection with numerical cognition (studies of Dehaene, Butterworth, Carey, Overmann and others), but where I am particularly interested in the emergence and configuration of geometric thinking. This is a most traditional philosophical topic, but I do believe that it can be advanced in interesting ways by means of an interdisciplinary approach that integrates new historical insights with cognitive archaeology, new philosophical orientations with detailed work in cognitive science. Another aspect of the question is the relation of mathematical ideas to semiotic systems (forms of representation) and to human action; mathematical practices cannot exist without the semiotic component — diagrams, notations, formulas — and so the cognitive basis of math is not limited to the brain. Yet another is the historical study of the web of practices, how mathematical work is interlinked with basic human practices, with scientific work, with techniques. This includes the study of the interactions math-science, which for some time had been left out of the philosopher's agenda, and subsequently was posed in a biased way in terms of the 'problem of the applicability' of math. As I suggested above, and explained in more detail elsewhere, this formulation is ideologically laden.

Also central is the question of the objectivity of mathematical work. I agree with those who think that the main issue is the objectivity of mathematical discourse (or practices, or knowledge), not the problem of mathematical objects. How the assumption of mathematical objects derives from objective mathematical discourse, is rather well understood (Tait 2005, Parsons 2009), although more should be made to summarize and popularize this body of work.[(3)]

[(3)] I try to do such a thing in the introduction to a forthcoming paper (*Topoi*, special issue on 'Mathematical Practice and Social Ontology').

But the question how conceptual products can be objective, has not been explored in sufficient depth; an important part of my work focuses on this issue. My standpoint assumes that mathematics is first and foremost conceptual work, conceptual development of models and methods; it emphasizes the cognitive and pragmatist roots of mathematical knowledge; and it fully assumes its historicity, arguing that it doesn't conflict with intersubjective objectivity (see my 2016 book).

§ 3. — Meaning in mathematics.

Another big question is the problem of meaning in mathematics, a difficult question intimately connected with the issue of conceptual development and conceptual understanding.[4] Perhaps one should begin here remembering that, early in the 20th century, in the wake of new ideas about logic and empiricism, there was great distrust of notions such as 'meaning' and 'truth';[5] the idea was to replace such unclear concepts with a fully alternative account in terms of logical laws, formal languages and empirical results. This was reinforced by then-recent developments in math, which promoted modern axiomatics and the free interpretation of symbols (Hilbert's 'point' can be a number, or a beer mug). After all, the phenomenon of meaning was regarded as a 'subjective' one, which implied that it should be eliminated from a scientific, objective way of thinking. In direct contraposition to this modernist ideal, I claim that the phenomenon of meaning is essential to mathematics.

To raise the question, let me employ a quotation from Hermann Grassmann: "Mathematics in its most rigorous form, in its inexorable consistency, is in a position to preserve the students from the fashionable rule of ingenious phrases, and to turn them to the practice of rigorous logical thinking. This aim would not be attained, however, if one wanted to present just formula after formula, without conceptual development. One must rather have both — formal development and conceptual development, going hand in hand."[6] Grassmann might be reflecting here not only his philosophical ideas (reminiscent of Leibniz at this point), but perhaps

(4) Also linked with the question of depth, discussed in *Philosophia Mathematica*, Vol. 23:2, June 2015.

(5) See Gödel's reminiscences; a relevant work is Carnap's *Logical Syntax of Language*, 1934.

(6) The quote is from H. G. Grassmann's *Lehrbuch der Arithmetik*, Berlin, Verlag von T.C.F. Enslin, 1861, vi.

also his practical experience as a secondary-school teacher; many disciples fail to see any relevance or sense in mathematical formulas, as a result of their inability to join the formal manipulations with conceptual counterparts. This is often the source of failures in the formal manipulations themselves.

We make sense of mathematical symbols by associating meanings to them, but we are also able to disassociate senses from symbols, and find new counterparts, new senses. Notice however that reinterpretation only comes in after a long process of development, whether you think of it in historical terms or ontogenetically, as a matter of individual development.(7) One should not try to go too fast, presenting things "which have often cost several thousand years' labor ... as self-evident"; one should employ historical studies to promote "enlightenment" (Mach 1872, 1-2).

Meaning in math is to a large extent given by *use*, which explains why an epistemology of math will probably have to consider *practices*. But meaning is not simply identical with use, otherwise it would be impossible to reinterpret (and also to enrich the sense of some symbols). To describe the situation in just a few words, one might say that the symbolic plane incorporates enough freedom that meaning is intimately linked with use, and at the same time it can embed the uses into a richer context. This is connected with the fact that mathematical systems are not simply an abstraction from physical (or mental) reality, but in effect incorporate hypothetical assumptions that create or constitute richer structures (that may well be that the case with structures endowed with continuity, see below). The bottom-up movement of reflections on concrete action and practices, is joined with the top-down ingredient of hypothetical postulates. It is thus that mathematicians can consider 'hypothetical states of affairs' and study what necessary conclusions follow from the assumptions (axioms) characterizing such structures.(8)

Many philosophers have recurred to model theory (formal semantics) in their attempts to get a solid grip on the slippery issue of meaning. Let me confess frankly that I don't trust in mathematical solutions to this, I don't look for the answer in formal semantics;

(7) It's also a familiar phenomenon that people often feel uncomfortable if something other than the customary symbol is employed in formulas (e.g., Dedekind's $\sqsubset$ instead of $\subseteq$ for inclusion, or Peano's $\supset$ instead of $\rightarrow$ for the conditional).

(8) This is C.S. Peirce's apt way of characterizing his (Riemannian) understanding of the conceptual work of mathematics. See Carter 2014.

it should not be found by developing *more* mathematics, but by analyzing what *underlies* math. I believe that key insights have to be obtained at a basic level, studying basic cognition, language and semiotics, and various kinds of practices, among which are mathematical practices — but also 'technical' practices and scientific practices.

I have proposed a thesis of the *complementarity* of formula and meaning in the practice of mathematics, none of the two poles being reducible to the other (see Ferreirós 2016, chap. 4). Admittedly, all of this is rather obscure, and my own work so far is only exploratory.

§ 4. — A hotchpotch of further remarks.

Of course, there have been other topics that guided some of my research. Initially e.g. the question of logic and its relations to math was quite relevant to me,(9) and in connection with it the issue of how properly to understand second-order logic. Also connected with this line of work is the basic notion of arbitrary infinite sets (and functions), which I have tried to clarify and bring to the fore. Modern set theory involves not only the study of infinite sets, but the emphatic consideration of *arbitrary* infinite sets (it rejects any constructivist or predicativist restriction on sets), which is essential to the usual understanding of power sets. That's often called the *quasi-combinatorial* viewpoint, and I argue that such ideas and assumptions properly belong to mathematics, not to logic.(10)

Some of the questions I have tried to study are quite inevitable for a philosopher who takes math seriously, like the conundrum of continuity: as Riemann called it, the "antinomy of the discrete and the continuous." This includes the question whether natural phenomena and their motions can be understood without continuity assumptions.

I should also acknowledge that, ever since I started my Ph.D. guided by Javier Ordóñez (Madrid), the conviction that history of science is indispensable for the philosophy of science grew stronger

(9) Logicism and related matters, including what I recently called the "dark heritage" of logicism (see *Metatheoria* Vol. 10, 2020, Special Issue - Foundations of Mathematics, pp. 19-30).

(10) Nevertheless, many logicians, being used to the idea by long years of training (as a result of widespread promotion by many relevant names), want to include them under second-order logic, or even plural logic.

and stronger. Ordóñez convinced me to work on the history of math, rather than on a more purely philosophical or foundational topic, and the effect upon my way of thinking and reflecting was noteworthy. I am convinced that the HPS approach is indispensable for attempts to deeply reflect on mathematics (see the introduction to Ferreirós & Gray 2006), and we should also bring mathematical issues to the debates about history of science and philosophy of science.

Let me mention, too, that over the years a number of topics in the connections between math and culture have occupied me quite a lot: for instance the Enlightenment, neohumanism in the German-speaking area during the nineteenth century, or the influence of philosophers on some key figures in the history of math.

Thinking more about questions that guide other researchers, and topics for the future, one may expect that the question of reorienting math's role in human practical life and technology may become prominent. Along the twentieth century, the spirit of mathematical research has been increasingly affected by technological orientations and economic life. As everyone knows, our culture measures everything by its economic impact — sadly, I add. The high spirit of pure math that was so prominent in the first part of the twentieth century, is different from the reigning spirit in the 21st century. Could math help advance towards global cooling and adaptation to climate change? Could there develop a movement of mathematics for peace and equality? In principle yes, but one remains skeptical thinking that the main influences come from practical life, from the economy and 'the markets'.

§ 5. — Back to mathematics and philosophy.

Mathematics does not bring to philosophy a special method that may solve its problems — I believe Kant was right in that. And even less does philosophy to math; it may bring some enlightenment, helping dispel some fogs, but not tools to help in concrete investigations. Nevertheless mathematics does have its philosophical conundrums, and so a conscientious mathematician will end up pondering philosophical questions. An example: what is the best balance between constructive methods and abstract-structural ones? how does it depend on the problem under study?

Is the current default foundation the most adequate one? In particular, does full-blown set theory create pseudoproblems that one would rather avoid? This writer, for one, believes that the Continuum Problem is not a well-defined mathematical question,(11) i.e., it's asking for full precision and control at a level where that cannot to be had.(12) Perhaps it would be wiser to regard the 'class' of real numbers as a virtual totality, not a fully determinate set. Perhaps one should devote one's efforts to elaborating mathematical tools to deal with systems that may or may not be continuous, instead of assuming that the continuum is fully understood and trying to build a most complex superstructure on top of it. Perhaps this is, after all, something that many scientists-mathematicians have been trying to do in the past and today.(13)

As the reader may intuit from those remarks, my position concerning mathematics and its foundations is pluralistic. If math is mostly conceptual work, as I believe it is; if it's partially dependent on hypothetical assumptions; then one should not aim at having a single all-encompassing framework. Let's consider instead a plurality of frameworks or theoretical systems, and let's employ them as best we can in the (piecemeal) exploration of reality that we call science.

§ 6. — References.

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(11) Even accepting Gödel's argument that proofs of independence from ZFC do not have to constitute a solution of the question. See Feferman's 'The Continuum Hypothesis is neither a definite mathematical problem nor a definite logical problem' (still unpublished), a revised version of his 2011 EFI lecture at Harvard.

(12) In the question whether $2^{\aleph_0}$ is or is not $\aleph_1$, we are asking whether one partially defined domain is bijectable with another partially defined domain (the powerset operation is eminently "amorphous", as prominent set theorist R. Jensen said). Yes or no? — Indefinite: we should allow truth-value gaps.

(13) See e.g. the recent lecture by M. Atiyah at the 5th Heidelberg Laureate Forum, 2017, 'The discrete and the continuous from James Clerk Maxwell to Alan Turing'; or consider the ideas of R. Penrose.

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