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Logic, Mathematics, and Philosophy

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As is common in political science, physics, and many other disciplines in relation to the relevant subfields of philosophy (political theory, philosophy of physics and so on), the more reflective parts of mathematics overlap considerably with the subfield of philosophy that is concerned with mathematics, with the philosophy of mathematics. It can seem, then, that the relationship between philosophy and mathematics is the same as that between philosophy and other academic domains, the subfield of the one essentially like the very theoretical, abstract, and foundational end of the other. The discipline of logic in relation to mathematics and philosophy provides a different perspective. Although logic was traditionally a topic for philosophers, it is now also a field of mathematics: mathematical logic is a study of patterns of reasoning that uses (or at least can use) distinctively mathematical methods. Such study does not exhaust the purview of logic, but even logic more broadly conceived has a special relationship to mathematics. From its first beginnings in Aristotle and still today, logic looks to mathematics as a paradigm of human reasoning.

History reveals further connections. Not only is logic transformed over the course of history, and this in three stages, beginning with Aristotle's term logic, through Kant's quantificational logic, to Frege's *Begriffsschrift* or concept-script; the history and practice of mathematics likewise has progressed in three stages, beginning with ancient Greek diagrammatic practice, through the practice of constructive algebraic problem solving made possible by the symbolic language Descartes introduced in 1637, to the sort of deductive reasoning from defined concepts that has been the norm since the nineteenth century. And in every case, it is mathematics that leads the way: profound transformations in mathematical practice ground and catalyze equally profound transformations in logic.

Those developments in logic enable in turn radically new forms of philosophical practice. As I show in *Realizing Reason*, Descartes' new mathematical practice with symbols provides the foundation not only for Newton's transformations in fundamental physics but also for Kant's radically new critical philosophy, at the heart of which were profound advances in logic. Similarly, the image we inherit from Kant of logic as a purely formal discipline without content or truth is superseded with developments in logic due to Frege following developments in mathematical practice over the course of the nineteenth century. These logical developments due to Frege, although misunderstood for almost the whole of the twentieth century, ground in turn advances in core areas of philosophy. And they hold out promise of advancing work in mathematics as well. It is this history that is the focus here, in particular, the logical developments that follow the two revolutions in mathematical practice, the first in the seventeenth century and the second in the nineteenth century, in light of the philosophical problem of how knowledge in mathematics is possible at all—developments that have significant ramifications, in turn, for mathematical practice. The movement we trace, then, is from mathematics to logic, to philosophy, and then back again to mathematics, first, in the seventeenth and eighteenth centuries, and then again in the nineteenth century until today.(1)

Ancient Greek mathematics is a science of objects, geometric shapes such as circles and spheres, and numbers conceived as collections of units. The philosophical problem is to understand, first, the nature of these objects, and also how it is that we are able to discover the timeless necessary truths about them that we do in mathematics. Plato had a nice account of mathematical objects: they are purely intelligible entities that exist outside the ever-changing realm of becoming of everyday life. His problem was to account for our cognitive access to such objects. Aristotle denied that there are any such objects; according to him, there are only everyday perceptible objects that are regarded in a peculiar, distinctively mathematical way in the mathematician's practice. What Aristotle could not explain was the distinctive character of mathematical truth, the fact that it is, unlike empirical truth, strictly necessary and a priori. Plato had a nice account of the metaphysics of mathematics but one that made the epistemology wholly intractable. Aristotle had instead a plausible epistemology, but no

(1) Further elaboration and defense of many of the themes outlined here can be found in my *Realizing Reason*.

adequate account of mathematical truth. This problem of truth and knowledge in mathematics, made famous for us by Paul Benacerraf in “Mathematical Truth”, is as old as philosophy itself.

In 1637, the practice of mathematics was transformed with the introduction of Descartes’ symbolic language within which to construct algebraically solutions to problems. What had seemed to the ancient Greeks to be truths about objects are now to be seen as directly about relations objects can stand in. Mathematics becomes the study of functions, curves in Cartesian space (another major conceptual advance) that can be expressed as equations in Descartes’ newly devised written language with its two essentially different sorts of letters, those for unknowns and those for the known parameters of a problem. A century on, these two sorts of letters would come to be reflected in Kant’s transformed understanding of logic as grounded in a logical distinction between referring (Kantian intuitions) and predication (Kantian concepts). Aristotle’s logical notion of a term, which is at once referential, a name for things, and predicative, a means of characterizing things in categorical sentences, was now to be seen as a *confusion* of two essentially different logical functions. Because Kantian concepts are purely predicative, and because generalities—which in Aristotle’s logic directly concern the objects that are referred to by the subject term—involve only concepts, Kant needed also to introduce an essentially new logical notion, that of a quantifier by which to relate the concepts in a general judgment to objects relative to which the judgment is true or false. Although only monadic, and not yet fully symbolic, Kant’s logic is a quantificational logic, the first such logic in all history.[(2)]

Kant also had a nice account of truth and knowledge in the practice of mathematics—not only the diagrammatic practice of the ancient Greeks but also and most especially the constructive algebraic practices of seventeenth- and eighteenth-century mathematics—in terms of the idea of the construction of concepts in pure intuition. Mathematical truths, on Kant’s account, are about empirical objects but only as to their a priori form, and we can know those truths because we construct a priori such objects in the course of our mathematical reasoning. No sooner had Kant

[(2)]That it is with Kant that central themes of quantificational logic make their first appearance in history is pointed out by Tiles (2004) in her discussion of Kant’s Transcendental Logic. This is also a crucial lesson, independently arrived at and defended, of my *Realizing Reason*.

formulated the account than mathematicians began to prove him wrong. Mathematics, at least as it came to be practiced over the course of the nineteenth century, is not grounded in pure intuition by way of the construction of concepts but directly in concepts. Mathematical truths concern (mathematical) concepts, and are known, as Bolzano illustrates in his 1817, purely conceptual, deductive proof of the intermediate-value theorem, not through constructions but through deductive reasoning from explicitly formulated definitions of concepts.

By the end of the nineteenth century, the deductive proof of theorems from defined concepts had become the norm in mathematical practice. Because this practice seemed to involve a discursive use of reason—as such a use contrasts, in Kant's philosophy, with an intuitive use through the construction of concepts, the use that Kant had held is that to which reason is put in mathematical practice—Kant's solution to the problem of truth and knowledge in mathematics came to seem utterly beside the point. The problem of truth and knowledge in mathematics had not after all been solved. Indeed, it now seemed more intractable than ever: how, by deductive reasoning from concepts, might one discover substantial mathematical truths? If Kant was right about the formality and sterility of purely logical, deductive reasoning, the idea of mathematical proof as at once ampliative and strictly deductive was simply, and obviously, incoherent. Frege determined that Kant was not right.

At the heart of standard quantificational logic, the fundamental idea of which was (again) developed by Kant, is the logical distinction of referring and predicative expressions. It is this distinction—together with the notion of a quantifier that is required for the completion of logic so conceived—that distinguishes modern logic from Aristotle's term logic. At the heart of Frege's logic is a further logical distinction. As from the perspective of quantificational logic, Aristotle conflates the logical functions of referring and predicating, so from the perspective of Frege's logic, the quantificational (Kantian) distinction of referring and predicative expressions conflates two logically different distinctions, that of *Sinn* and *Bedeutung* with that of (Fregean) concept and (Fregean) object. The Kantian conception of a concept confuses the notion of a Fregean concept, understood as a function mapping arguments onto truth values, with that of cognitive significance, that is, Fregean sense through which one thinks anything objective at all.

Pace Kant, it is not through concepts, predicates of possible judgments, that objects are thought, but instead through Fregean senses. And the Kantian notion of an object similarly confuses objectivity with relation to an object. Although in Kant's logic, all content and all truth lie in relations to objects, so that without relation to any object there is simply no content, no truth, this, Frege shows, is wrong. Concepts, the *Bedeutungen* of predicates, are perfectly objective despite not being objects. What the mathematician discovers, on Frege's account of mathematical practice, are necessary relations among mathematical concepts. Truth, at least in mathematics, does not lie in relation to objects. There are no mathematical objects.[(3)]

Frege claims in his 1884 *Grundlagen*, section 91, that he had shown in Part III of his 1879 logic how a strictly deductive proof can be ampliative, that his proof of theorem 133 in *Begriffsschrift*, which is strictly deductive from explicitly defined concepts, extends our knowledge. As Frege (1884, 104) puts it, "from this proof it can be seen that propositions which extend our knowledge can have analytic judgements for their content." In Chapters Seven and Eight of *Realizing Reason* I aim to show in detail why we should take Frege at his word here. Fortunately, we do not need to study a long *Begriffsschrift* proof to get the essential idea as needed for purposes here. Although it could only be discovered after millennia of inquiry together with profound mathematical and logical advances, the phenomenon in question is manifest, in retrospect, even in the very simplest mathematics, and indeed is a variant of what Kant already understood about the constructive nature of mathematical practice.

Frege's discovery of the distinction, conflated in Kant's logic, between *Sinn* or cognitive significance and concepts, on the one hand, and objects and objectivity, on the other, is a discovery about how language, most obviously and immediately mathematical language, works. It applies to anything that is properly thought of as a language, but is clearest in a specially devised written system of signs suitable for reasoning in mathematics. Frege's discovery, as applied to the case of mathematical language in particular, is that

(3) Frege himself thought that there had to be mathematical objects for numbers to be. He was wrong about that. Mathematics needs only *concepts* of numbers. And in retrospect, this seems obvious given that it is on the basis of concepts that one reasons. Nothing follows from something's being the object it is, but only from the properties it has, that is, from concepts under which it falls. Any and all reference to the object simply falls away; objects are irrelevant to mathematical practice.

the primitive signs of the language—which in the case of mathematics is (again) a specially devised system of written signs within which to reason in mathematics—function first and foremost to express Fregean sense, *Sinn*. Such signs do not designate, at least not as such. It is only when those signs are combined in some context of use that they designate or have *Bedeutung*. Imagine, then, an absurdly simple mathematical language, a stroke language with only one primitive sign, the stroke, which is to be used in the formation of complex signs that designate numbers and in (absurdly simple) reasoning about numbers. (Although we discover over the course of the history of mathematics that mathematics has no objects, we can assume for the purposes of this very simple example that there are mathematical objects, in particular, numbers.) We begin, then, by forming complex signs for, say, the numbers seven and five using our one primitive sign: /////// /////. These two complex signs are not to be read (seen) simply as collections of units. They are *meaningful* signs of a particular *mathematical language* and are to be read as such, the first as a sign designating the number seven, that particular unitary mathematical object, and the second designating the number five, a different mathematical object, both through the display of what it is to be such numbers, namely, on the relevant conception, certain multiplicities. Because the contents of these numbers as they matter to reasoning in the system of signs are displayed in the signs that designate those numbers, we can manipulate the signs according to antecedently stated rules to discover something new. In particular, in this case, we can construct from the two original signs a sign for the number twelve, demonstrating thereby that seven plus five equals twelve, a properly mathematical result. (Notice that simply combining the strokes and counting the resultant collection would *not* constitute a piece of mathematical reasoning.) Our stroke language, simply though it may be, is at once a Leibnizian lingua and a calculus ratiocinator. Complex signs in the language display the contents of the mathematical entities those signs designate; the language is a Leibnizian lingua. And they do so in a way enabling rigorous, rule-governed reasoning on the basis of those contents *in* the system of signs; the language is at the same time a calculus for reasoning.

Reasoning in Frege's mathematical language *Begriffsschrift*, although much more sophisticated, is essentially the same, at least in this respect. Using primitive signs that in themselves only express Fregean senses, one constructs complex signs for mathematical concepts of interest, signs that at once designate those concepts and display their contents as they matter to inference. Then, following antecedently stated rules, one manipulates those expressions, sometimes individually, more interestingly in combinations, to yield, ultimately, some new result, one that reveals a logical relation among the concepts with which one began. As Frege himself explicitly saw, such reasoning is, or at least can be, ampliative, a real extension of our knowledge, despite being strictly deductive. And this is possible, I argue in Chapter Eight of *Realizing Reason*, because not all deduction, that is to say, necessary, truth-preserving reasoning, is *logically* deductive. Although most steps in a chain of mathematical reasoning are purely logical, some can be—as they are in Frege's proof of theorem 133 in Part III of *Begriffsschrift*—licensed by rules of inference derived from definitions, in the case of Frege's proof of theorem 133, from Frege's definition of following in a sequence. Such steps, because they are deductive without being licensed by logic alone, are ampliative. They extend our knowledge.

Frege's advances in logic enable us to understand how a purely deductive proof such as, for example, Bolzano's 1817 proof of the intermediate-value theorem, or proofs of significant theorems in abstract algebra, can be ampliative. Indeed, they enable us to distinguish between *three* different degrees of ampliativity in a mathematical proof, and this along three different dimensions.

First, we know already from Kant that judgments that are ampliative in mathematics are synthetic a priori; they are necessary but not logically necessary. What we are interested in, then, is a distinctive sort of unity in mathematics, what I call an intelligible unity to distinguish it both from an essential unity, a real whole with no real parts, and from an accidental unity, which has real parts but is not a real whole. An essential unity in this context is to be understood as something that is logically necessary, as an analytic truth is on Kant's account. Because in an analytic judgment as Kant understands it, the predicate is contained already in the concept of the subject, the judgment itself is a real whole without any real parts. To try to separate out the parts, by denying the predicate of the subject concept, is to fall into a logical contradiction. At the other end of the spectrum are accidental unities. An a posteriori judgment is such a unity insofar as it has real parts, the subject concept and predicate, but because these two components of an a posteriori judgment stand in no real, or necessary, relation, the judgment as a whole lacks any real unity. The two concepts, subject concept and predicate, happen to belong together in virtue of certain contingent facts about empirical objects, but they just as easily could be without any relation at all. An a posteriori judgment is an accidental unity; it has real parts but is not itself a real whole. Synthetic a priori judgments are, of course, the interesting case. Because they are a priori, that is, necessary and strictly universal, such judgments are real wholes. But they also have real parts insofar as there is no contradiction in denying the predicate of the subject concept. The two concepts in a synthetic a priori judgment are logically distinct, and yet they are necessarily related. A synthetic a priori judgment is, then, an *intelligible* unity; it is a real whole, unlike an accidental unity such as an a posteriori judgment, but it also has real parts, unlike an essential unity or analytic judgment. Now we need to apply this idea of an intelligible unity, not only to judgments, that is, to the theorems of mathematics, but also to its defined concepts, and to its chains of reasoning.

Following Frege, we distinguish, first, between fruitful and other definitions, where (as Frege thinks of it) a fruitful definition draws new boundary lines. By contrast with a definition that does not draw new lines—such as Frege's definition of belonging to a sequence, which, like the definition of less-than-or-equal-to, is merely disjunctive, merely an accidental unity of parts introducing nothing new—definitions that draw new lines involve both the conditional stroke and Frege's concavity with widest scope.[(4)] Such definitions are intelligible unities insofar as they clearly have real parts, as indicated by the primitive signs they involve, but are also real wholes. Again, in Frege's imagery, they draw new boundary lines. (Primitive signs, because they are undefined, can be thought of as essential unities; they are wholes without parts.) Proofs that combine content from two or more fruitful definitions, for instance, those that show that one such defined concept is logically related to another by subordination, will have conclusions that are themselves intelligible unities. Such conclusions are, in Kant's terminology, synthetic a priori insofar as the predicate concept is not contained in the subject concept but can be shown through the course of reasoning to be necessarily related to it. Theorems that

[(4)]Contrary to the received view, Frege's concavity is not a notational variant of the universal quantifier of standard logic, as is indicated already by the fact that it serves in the formulation of laws. See my *Frege's Logic* for details.

merely draw out some consequence of this or that definition, even one that is fruitful, are instead analytic. Nothing new emerges in that case.

We have just seen that the signs designating concepts in a mathematical language such as Frege's *Begriffsschrift* can be divided into three sorts. Primitive signs are essential unities that designate concepts directly; they have no real parts. Some defined signs, such as the sign for less-than-or-equal-to, function merely as abbreviations; they are accidental unities that introduce nothing new. The last and most interesting signs are those for concepts that have fruitful definitions, definitions that draw new lines. Fruitful definitions reveal the concepts they define as intelligible unities, real wholes of real parts: the concepts are definable but they are nonetheless significant concepts in their own right. They are not reducible to their parts in combination. Concepts that have fruitful definitions thus have a kind of unity and integrity that is lacking in concepts designated by mere abbreviations for complexes of signs. They constitute the subject matter of significant domains of mathematics and are the material basis for significant mathematical theorems.

Among mathematical signs, primitive and defined, there are essential unities, namely, primitive signs, accidental unities, defined signs that function merely as abbreviations, and intelligible unities, defined signs for the concepts about which one reasons in mathematics. And as we know, mathematical theorems likewise can be divided into sorts. As most would agree, there are no merely accidentally unified judgments in mathematics, no a posteriori judgments; all truths of mathematics are necessary. But among the necessary truths of mathematics there are both analytic judgments, that is, essential unities, the negations of which are logical contradictions, and synthetic a priori judgments, that is, intelligible unities, real wholes of real parts, judgments that are necessary but not logically necessary. Now we need to think about different sorts of mathematical reasoning, differences among chains of reasoning in mathematical proof.

First and most obviously, no mathematical chain of reasoning is accidental. Mathematical reasoning—by contrast with, say, reasoning (inductively) from cases to a generality that may or may not in fact be true—is necessary reasoning: if the premises are true then the conclusion must be true as well. And there are clearly proofs, chains of reasoning in mathematics that are essential unities, that are logically necessary throughout. In such a proof, each step is licensed by a rule of logic; and because it is, the conclusion is analytically contained in the definitions with which one begins, even in the case in which the conclusion itself is a synthetic a priori judgment, that is to say, an intelligible unity. But there are also mathematical proofs, chains of mathematical reasoning, that seem clearly to be not analytic in this way but are, it seems, ampliative, real extensions of our knowledge. (Again, Frege claimed that his proof of theorem 133 in Part III of *Begriffsschrift* is such a chain of mathematical reasoning.) And if there are such chains of reasoning in mathematics, they must have steps that are necessary but not logically necessary, not instances of purely logical inferences. Such steps must be instead materially valid, valid in light of the meanings of the non-logical concepts involved.[5]

Very often in mathematics, deductive proofs of theorems, that is, not only the conclusions but the chains of reasoning, seem clearly to constitute real extensions of our knowledge. We are trying to understand how they do; and for this, we need a distinction that Frege did not draw, though it is evident in his inferential practice. We need to distinguish between those fruitful definitions that enable proofs of theorems that are synthetic a priori but involve only purely logical reasoning, and those fruitful definitions that enable proofs not only of theorems that are synthetic a priori but that are themselves ampliative because involving those definitions *in* the steps of reasoning. In these latter sorts of cases, definitions provide not only premises from which to reason but also principles according to which to reason. The reasoning in such cases is deductive, necessary, since in such cases if the premises are true then the conclusion must also be true, but the reasoning is not logically deductive, not by logic alone. The definition is essential to the reasoning insofar as, assuming we have correctly defined the relevant mathematical concept, the step of reasoning is valid *as* licensed by that definition, despite not being logically valid. Because and insofar as without that definition there is no proof, no path of strictly deductive reasoning from the definitions to the theorem, not only the definition with which one begins, and the theorem that is proved, but even the reasoning itself constitutes an intelligible unity, a real whole of real parts.[6]

[5] On the notion of material valid inference as it contrasts with logically valid inference, see Sellars (1953).

[6] See Chapters Seven and Eight of *Realizing Reason* for further development and defense of this claim.

We have just seen that mathematical definitions can be fruitful (in Frege's sense) in two essentially different ways. All fruitful definitions are fruitful in themselves insofar as they draw new lines; they are real unities of parts that can provide the grounds for synthetic a priori judgments of mathematics. But some fruitful definitions also enable fruitful, that is, ampliative, steps of reasoning within proofs. They enable deductive steps of reasoning that are not strictly logical but are nonetheless necessary, strictly deductive insofar as it is impossible for the premises to be true and the conclusion false. Such a definition thus can be used to license an inference that is not otherwise available to be drawn. Frege's definition of following in a sequence is just such a definition: assuming that it gets the content of the relevant concept right, the inferences it licenses are valid, but they are not logically value; they do not follow by logic alone. Chains of reasoning that depend on such steps of inference, such as that involved in Frege's proof of theorem 133, not only start from fruitful definitions and have conclusions that are synthetic a priori; the reasoning itself is, in its way, synthetic a priori. The relationship of the conclusion to the definitions with which the reasoning begins is necessary but not logically necessary, and hence the proof as a whole is not only a real whole, as is the case in any strictly logical reasoning, but also one that has real parts precisely because although the conclusion follows, it is not contained already in the premises, needing only strictly logical inferences to be make explicit. It is contained, to borrow once again imagery from Frege in *Grundlagen* (section 88), not as beams are contained in a house, that is, implicitly, already there only not obviously so, but as the plant is contained in the seed, that is, potentially. It is the course of reason, reasoning that uses not only rules of logic but also at least one rule of inference derived from a definition, that brings that potential to actuality.

Frege was a mathematician by training and by profession. He devised his logical language *Begriffsschrift* as a symbolic language within which to reason in mathematics. The immediate aim was to prove the truth of logicism, but the language he developed was, as he knew, of more general mathematical significance. As Descartes' symbolic language was a language within which to reason in the constructive algebraic tradition of eighteenth-century mathematics so Frege's language was to be a language within which to reason in the deduction-from-concepts tradition that emerged over the course of the nineteenth century. Because Frege's language was not understood, it was never used. Mathematicians' proofs as currently promulgated are reported in natural language, with a smattering here and there of logical signs serving as stand-ins for words of natural language, just as they were throughout the nineteenth century, that is, before the development of our modern, symbolic, quantificational logic. Unsurprisingly, at least in retrospect, the development of quantificational logic into a full logic of relations at the turn of the twentieth century did nothing to change the way mathematicians formulate and present their reasoning. Quantificational logic, which again, is merely Kantian, is utterly inadequate as a logic of mathematics as it has come to be practiced since the nineteenth century. What is needed is Frege's logic.

Frege's logic was enabled by nineteenth-century advances in mathematics as Kant's logic was enabled by seventeenth-century advances in mathematics. And as Kant's logic provided the foundation for Kant's radically new practice in philosophy, his critical turn, so Frege's logic provides the foundation for a radically new philosophical practice. But as indicated, Kant did not only revolutionized the practice of philosophy; he also, in so doing, provided the first significant solution to the problem of truth and knowledge in mathematics. Frege's logic as providing the foundations for a new philosophical practice likewise sheds new light on this venerable problem. Indeed, Frege's advances in logic enable fundamental advances in philosophy not only with respect to the problem of truth and knowledge in mathematics but, as I aim to show in some detail in *Realizing Reason*, as concerns reasoning, thought, and knowledge more generally. And such advances have the potential to advance in turn the practice of mathematics. Once having been apprised of how mathematics functions to extend our knowledge by deductive reasoning, and of how a specially devised written mathematical language can display such reasoning, mathematicians may well discover that this image of mathematics bequeathed to us by Frege not only advances the teaching and learning of mathematics, by putting the reasoning before students' eyes just as earlier mathematical languages did, but also alters assessments of what is and is not a tractable problem in mathematics. The great benefit of Frege's notation, to one who knows how properly to read it, is the way it enables setting out in a two-dimensional array the contents of mathematical concepts as they matter to inference. Who knows what new mathematics may be enabled by mathematicians being able finally to see, literally to see, the contents of the concepts with which they are concerned?

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