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Towards Mathematical Criticism

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§ 1. – Introduction.

Often, in the so-called "Philosophy of Mathematics", Mathematics has been forgotten, and replaced by linguistic and speculative considerations, very distant from the original creations of the mathematicians. Some alternative perspectives have tried to come back to Mathematics (*e.g.* a "Synthetic" philosophy of mathematics, a philosophy of "Mathematical Practice"), but a fair assessment shows that a deeper understanding of *mathematical creativity*, looking with further precision into the very mathematical fabric, is still needed.

In that order, a simple Pascalian reason — "Mathematical Criticism should be the same to Mathematics as Literary / Artistic / Musical Criticism is to Literature / Art / Music" (see *Figure 1*) — drives our search. One of the main outputs of this approach is that what we may call (A) "Mathematical Criticism" turns out to be very far apart from the standard paths taken in normal (B) "Philosophy of Mathematics". In that distinction, we think that perspective (A) serves Mathematics in a more *full and faithful* way, which may be useful for the discipline.

Our article is divided in three sections. *Section 1* studies the establishment of a new field of inquiry, "Mathematical Criticism", akin to Literary Criticism, Art Criticism, or Musical Criticism, in which the main focuses of *critics* correspond to (*i*) precise descriptions of the *creative* works involved, (*ii*) thorough *internal and external* assessments of the works described, (*iii*) *calibrations* of their mathematical, philosophical, and cultural irradiations. *Section 3* offers a case study, around Galois and Riemann, comparing method and substance in the "Analytic Philosophy of Mathematics" with the "Mathematical Criticism" envisioned in the previous section. *Section 4* reviews some examples of "pioneers" of Mathematical

Criticism (*e.g.* Shaw, Lautman, De Lorenzo), often subsumed under a larger and more vague adscription to the "Philosophy of Mathematics", and presents our new RTSK models where Mathematical Criticism acquires a natural life.

$$\frac{\textbf{Mathematics}}{\textbf{Mathematical Criticism}} \equiv \frac{\textbf{Literature / Art / Music}}{\textbf{Literary / Artistic / Musical Criticism}}$$

Figure 1. — Towards Mathematical Criticism

§ 2. — Acknowledging the Need of Mathematical Criticism.

By their very practice, Literary Criticism, Art Criticism, and Musical Criticism, *delve into the creative works* of writers, artists, composers. In that sense, it is entirely impossible to understand Modern and Contemporary literature, art, or music, without entering in detail into the works of Proust, Manet, or Beethoven, just to mention some striking examples. In the same way, Literary Criticism stands on the towering figure of Walter Benjamin (and his Proust *critique*, for example), Art Criticism rests on Aby Warburg (and his Manet dissections), and Musical Criticism lies over Heinrich Schenker (and his Beethoven studies). Both (1) a fine attention to the works involved (*La Recherche*, *Le Dejeuner*, or the *Quartets*) and (2) their careful description-assessment-calibration are needed in order *to really sense and understand* literature, art, music. Following Pascal's *dictum* — "The heart has its reasons that reason knows nothing about" — we need to take into account both a "sensorial" aspect (heart, in Spanish "corazón" = "co-razón", the exact dual of reason) and a "rational" aspect, to understand a field of knowledge. Particularly, if that field tends to be very *creative*, we may need consistently a back-and-forth between perspectives (1) and (2) to capture *fully and faithfully* the given field (see *Figure 2*; the back-and-forth may be represented by a *functor*, which captures contexts, obstructions, and transferences).

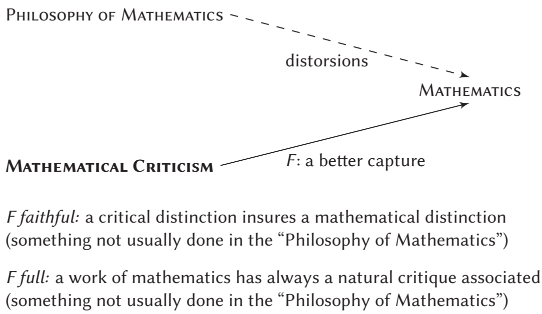

FIGURE 2. — A full and faithful functor F

If we follow [Francastel 1965], who reckoned that Art and Mathematics are the major poles of creativity in human thought, Mathematics would then be very much enriched by a Mathematical Criticism where the extrapolations of (1) and (2) to Mathematics would be mandatory.

The many *stratifications* of a creative work were beautifully captured in Gilles Deleuze's Foucault course (Vincennes, 1985-86). In a recent edited transcription of the course[(1)], Deleuze presents a deep understanding of cultural complexity, around Melvillean metaphors[(2)]. Building on *Moby-Dick; or, The Whale* (1851) and *Pierre; or, the Ambiguities* (1852), Deleuze constructs a "Whaling Line" diagram, in which he shows a profound understanding of *traversing culture*, both around natural and scientific languages. In a back-and-forth between sight and language, Deleuze describes "thick" strata where some statements are inscribed, to be further "opened" and studied through descents and ascents in an "oceanic zone" of knowledge. There, a "line of the outside" between the Self and the World is captured through its "velocity" and "foldings", producing for each of us a complex horizon of learning (see *Figure 3*)[(3)].

Deleuze evokes then "a mathematical genius, Galois, who made sorts of proofs with ellipses, deviations, precipitations, fulgurations"[(4)]. Galois is brought up as an example of "real mathematical

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(1) See [Deleuze 1985-86]. The Spanish transcription has not yet even appeared in French.

(2) Lesson of 25 May 1986. *Ibidem*, pp. 173-204.

(3) The diagram is ours, based on the transcription of Deleuze's lesson, while he draws on the blackboard. The audio was taped, but we do not know of a video recording which captured the blackboard.

(4) *Ibidem*, p. 194.

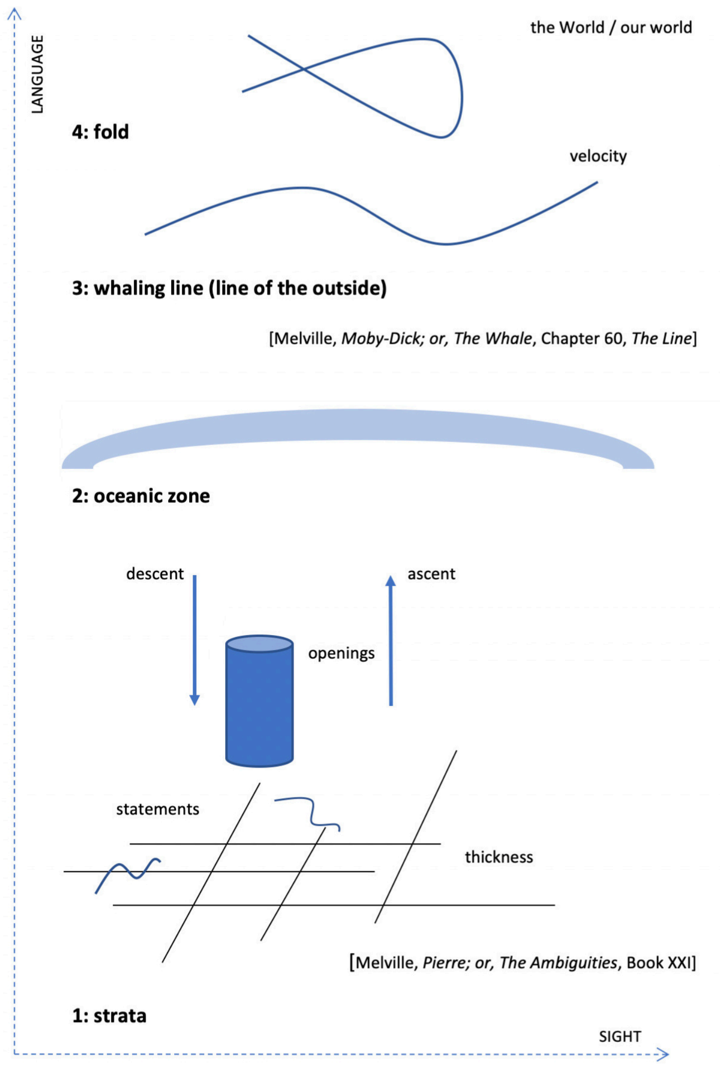

FIGURE 3. — Deleuze's Last Lesson – Foucault Course (20 May 1986)

creation", where the mathematician plunges in a series of "fulgurations, with blanks, deviations, etc."[5] Immersing in Black, Deep Waters — well beyond the futile "clearness" sought by Analytic Philosophy —, Deleuze studies the "velocity of thought" captured

[5] *Ibidem*, p. 193.

by mediations, deviations, ramifications. In that *mud* of knowledge, mathematical creation responds naturally to literary, artistic, or musical creation, and many layers above deep bottoms represent the search for a dynamic, ever fluctuating knowledge.

In Deleuze's diagram, basic strata — either literary, artistic, musical, or mathematical — support ascents and descents in open, imaginary zones, where the lines of the Self and the World entangle, to introduce the fundamental dialectics which guide our understanding. The emerging creative Work — either literary, artistic, musical, or mathematical — is then looked at by the Critic, who can describe and uncover the raw forces which shape the written, plastic, or symbolic cultural production observed. But, contrary to Art Criticism, Musical Criticism, or Literary Criticism, well established cultural practices, *Mathematical Criticism* has never surfaced as a field on its own, and has usually been confused with fragments of the Philosophy of Mathematics (see *Figure 4*).

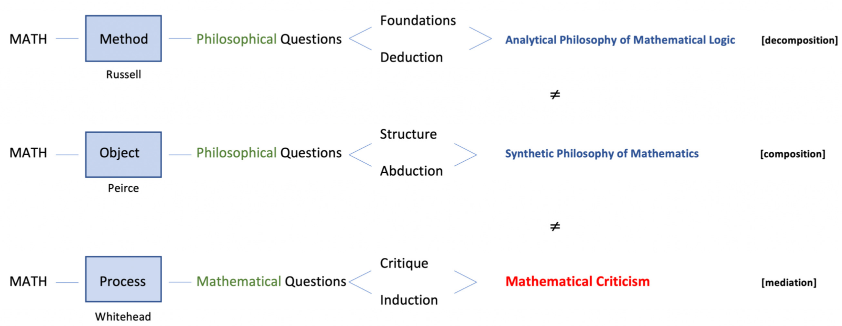

FIGURE 4. — Towards Mathematical Criticism

As the Art Critic deals with artistic questions, the Literary Critic deals with literary questions, and the Musical Critic deals with musical questions, the Mathematical Critic has to deal with *mathematical* questions. A *real knowledge of mathematics* is thus necessary, a knowledge *at large* which cannot be reduced, for instance, to philosophical questions about the language of logic and set theory (= "Analytic Philosophy of Mathematical Logic and Set Theory", an interesting but extremely restricted field, often confused with the "Philosophy of Mathematics").

Mathematical Criticism can then be defined as the study of:

i. how mathematical thought is constructed (*description of processes*)

ii. how those constructions are entangled in ideal realms of possibilia (*critique*)

iii. how those entanglements are contrasted with the real (*induction*, in Peirce's sense).

Well beyond considerations of method and logic, mathematical creativity involves a complex web of "fulgurations, blanks, deviations", as noticed by Deleuze. In fact, [Poincaré 1908] and [Grothendieck 1983-86](6) explore with utmost detail the many zigzags and wanderings of their own inventions. In a sense, in those essays, the mathematician and the mathematical critic converge. But possibly the greatest example of a Mathematical Critic in the 20th century may be Albert Lautman(7) (see *Section 4* below), a towering figure not usually studied along the "normal" currents in the Philosophy of Mathematics. Lautman is the perfect example of the *Critic* — an outside observer: not a writer (*e.g.* Benjamin), not an artist (*e.g.* Warburg), not a composer (*e.g.* Schenker), not a mathematician — who describes, critiques, assesses, and synthesizes, all the complex variations and forces at work in a precise cultural context.

§ 3. – Analytic Philosophy versus Critics: Around Galois and Riemann.

In a similar way to the now popular "Philosophy of Mathematical Practice" — which has been growing in the last decade, but was preceded (and already superseded) by the work of Javier de Lorenzo (see *Section 4* below) — Mathematical Criticism emphasizes a *return to mathematics*, a way to fully and faithfully look and assess *mathematical doing* (De Lorenzo's battle horse). The crucial thing becomes, first, to capture mathematics, in order to,

(6) For a broad and, at the same time, very precise, guide to Grothendieck, see [Zalamea 2019].

(7) *Cf.* [Lautman 1935-42]. See also [Lautman 2011], the extended Spanish translation of Lautman's complete works, with many texts not present in the French edition.

subsequently, reflect upon it. "Mathematical Critics" points to this *double process* of knowing, and thus turns out to be closely situated to any other critical action (Literary Criticism, Art Criticism, Musical Criticism, Film Criticism, etc.) In fact, the task of the *Critic* is double: (1) to explore, know, feel, *live* the works under study, and only then (2) describe, calibrate, evaluate, *explain* them. The first stage is *mandatory*: without a deep immersion in a work, there are no critics. It is *not valid* to perform a *substitution* of the work by a discourse upon it and by n^{th}-order crossing references between specialists which no longer say anything about the original work. Unfortunately, this is the case of the "Analytic Philosophy of Mathematics", which has substituted, with no major shame, mathematical works — "mathematical doing" — by self-adulatory *clique* references and by linguistic considerations, supposedly deductive, on logic (usually classic) and set theory (usually Cantorian), leaving aside all inventive power of mathematical thought (a Galois, a Riemann, a Grothendieck, for instance). The situation is as absurd as to pretend to analyze Modern art without ever mentioning Monet, Duchamp, or Picasso, Modern literature without studying Proust, Joyce, or Musil, or Contemporary Film without ever looking at Antonioni, Bergman, or Tarkovsky.

The "Analytic Philosophy of Mathematics" has restricted entirely its *substance* (reducing the ever growing complexity of mathematics to problems of foundations, around logics and sets) and has been essentially concerned with problems of *method*. The confusion is gigantic. As a good product [Shapiro 2005] of the school forcefully shows, some of the pillars of 19th century Modern mathematical thought — abstract algebra, complex variables, algebraic geometry, for example — are never mentioned(8). How can one talk about Mathematics, without ever referring to Mathematics? It seems crazy but it has been done: it is just a *miracle* — really a *mirage* — of the linguistic turn used in the "Analytic Philosophy of Mathematics", which must be forcefully denounced and confronted. Many years of clever and inconsequential discourses in the "Philosophy of Mathematics" have been accepted in the Academia without pondering some healthy *critical distance* on them. A fact

(8) The indexes (of both subjects and proper names) at the end of the volume — badly named "The Oxford Handbook of Philosophy of Mathematics and Logic", while a more humble "The Oxford Handbook of Anglo-Saxon Analytic Philosophy of Logic" would be much more indicative — refer only two pages (of the 833 in the volume) to Galois and Riemann; Grothendieck does not even appear.

about "Mathematical Criticism", if such a field turns out to be soundly developed[9], is that *mathematical substance* — a detailed attention to mathematical works, *forced* in step (1) above — will never disappear in its considerations, in sharp contrast with what has happened in the "Analytic Philosophy of Mathematics".

A good case study of the situation is the scant attention given in the "Analytic Philosophy of Mathematics" to Galois and/or Riemann. A quick search in the MSC database, shows, today, thousands of references to Galois and Riemann. Their role in *mathematical thought* is central to our understanding of the discrete and the continuous, of number and magnitude, of algebra and topology, of number theory and geometry. But the "Analytic Philosophy of Mathematics" *cannot see* them. On one hand, the structures they study, in the algebraic and differential realms, exceed logics and sets; on another hand, their very methods — Galois's *ambiguity* theory, Riemann's *uniformization* theorems — are inherently *vague*, in order to involve enough non-trivial automorphisms of the structures. An "analytical reduction" (around structures) and an "analytical clearness" (around methodology) are not only impossible, but would be *mathematically unsound*. In fact, the very *richness* of mathematics is directly correlated with what an analytical dissection throws to the garbage... In that sense, again, *if we want to understand mathematics*, Mathematical Criticism would be much more useful than a *blind* "Analytic Philosophy of Mathematics". Of course, if one wants *instead* to understand languages, logics and foundations, please turn to the "Analytic Philosophy of Logics and Sets", but *do not pretend to talk about Mathematics* in that restricted environment. We obtain thus a simple clarification of diverse trends in the "Philosophy of Mathematics" (see *Figures 2, 4* above).

A precise example of an important mathematical problem that an "Analytic Philosophy of Mathematics" approach would never capture is the eventual anticipation of Riemann surfaces in Galois's Last Manuscripts (*Sainte-Pélagie* Prison, July 1831 - March 1832; *Lettre Testamentaire*, May 29, 1832). Combining history and mathematical acumen, [Dieudonné 1962] anticipated the problem[10] as an instance of a (Lautmanian) mixture between the

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(9) We hope to advance this program in our next monograph [Zalamea 2023].

(10) "One can imagine that Galois was very close to the idea of a «Riemann surface» of an algebraic function, and that such an idea must have been fundamental in his researches on what he called the «Theory of Ambiguity»" [Dieudonné 1962, p. *v*].

elliptic-modular (Complex Variables) and the structural (Abstract Algebra). Galois had *seemed* to develop the algebraic architecture before the complex variable apparatus, as some traceable sources indicate: (*i*) an *additional* note (January 16, 1831) to the "Introduction" of the *Première Memoire*, where the young genius states that "the modular equations of elliptic functions cannot be solved by radicals"(11), underscoring an eventual application of a *prior* general algebraic theory to the specific case of the elliptic-modular case; (*ii*) the Sainte-Pélagie Manuscripts, with more than 70 pages on modular and elliptic *calculi* (by far the largest output of mathematical work done by Galois), advanced *after* his memoirs on non-solvability of equations. Nevertheless, a yet unnoticed indication prompts to the contrary, in support to Dieudonné's hypothesis. In fact, (*iii*) in the Prison Manuscript [176a] appears the cryptic addition of a date — "1er Mars 1827"(12) — which directs the Galois of 1831 to four years before. The annotation points to the beginnings of Galois's works (1827), but appears to be written at the end of his life (c. 1831). The situation is very surprising: could it be that there existed a *deep continuous* connection between 1827 and 1831, "invisible" to the eyes of the 1830 *superficial algebraic* fragments? The *Lettre Testamentaire*(13) offers a support to this perspective, unravelling *very similar methodologies* at work in Galois and Riemann: (*a*) visibility of the whole beyond the singular [8a], (*b*) canonical/iconical decompositions of equations [8b], (*c*) modular representations [9a, 9b], (*d*) uniformization of abelian integrals [10a, 10b], (*e*) final vision of an ambiguity/complexity theory [11a].

With this example, one sees that *without entering into the mathematics involved* (task (1) of the Critic above), one simply cannot expect to philosophize upon them (task (2)). *Without a deep immersion into mathematics*, many problems around content and method simply disappear of the panorama: an *inversion* between (*i*)-(*ii*) and (*iii*), that is, a wallowing that would change the *whole*

(11)Manuscript [2a], *cf.* [Galois 1962, p. 43].

(12)The date appears in a digital copy of Galois's Manuscripts at the Institut de France, in the middle of calculations around complex transformations ($\frac{ak+b}{ck+d}$), infinity points (∞) and the order of a normal group ($\frac{p+1}{2}$). The fact that the date *does not* appear in Azra and Bourgne's critical transcription [Galois 1962, pp. 325-327] is extremely indicative. The editors simply could not imagine an *inversion* of Galois's creative imagination, where the elliptic world could emerge *first*, and *then* lead the way to inventiveness in the algebraic world.

(13)Manuscripts [8a-11a], *cf.* [Galois 1962, pp. 173-185].

understanding of our discipline, cannot be stated, or even detected. In this sense, a Mathematical Criticism must be developed, as (1) a first and mandatory condition to (2) allow to reflect upon mathematics. We would then be able to (1) *plunge* into the technical constructions that support the entire life of the discipline, in order to afterwards (2) *emerge*, breathe, and think about our immersive experience. The (1) deep, muddy, obscure waters of the Ocean support our (2) superficial, clear, bright balance, along waves in the ocean shore (*cf.* Deleuze's diagram, *Figure 3* above, contrasting (1) *Moby-Dick* and (2) our "whaling lines").

§ 4. — Some Pioneers of Mathematical Criticism. The Models RTSK.

Mathematical Criticism has to share some *common processes* with other criticisms (Literary Criticism, Art Criticism, Musical Criticism, Film Criticism, etc.): First, a *direct observation* of the creative works involved; second, a *strategy to analyze, decant, and synthesize back* the observed constructions; third, the elaboration of a *systematic stratification* of perspectives, calibrations, and assessments of the works discussed; fourth, the construction of a *general, global* architecture of knowledge based on the particular, local case studies offered by the *critique*. In this *continuous back-and-forth* between analysis and synthesis, observation and speculation, localization and globalization, immersion and emergence, Mathematical Criticism should never choose a perspective over others, and should never attempt to produce impoverished reductions of a complex state of affairs.

This *multidimensionality* of mathematical thought — combining words and images, proofs and hypothesis, definitions and intuitions, theorems and errors, concrete types and universal archetypes — has often been heralded (*e.g.* in [Poincaré 1908]) as one of the *crucial richness* of the discipline. Three paradigmatic critical enterprises, deeply attentive to that multidimensionality, can be here mentioned: the works of James Byrnie Shaw (1866-1948), Albert Lautman (1908-1944), and Javier de Lorenzo (born 1939), among other great explorers of mathematical thought, not reckoned by the usual trends in the "Philosophy of Mathematics". Shaw's *Lectures on the Philosophy of Mathematics* [Shaw 1918] constitute a miracle of perspicacity in the middle of the First World

War[14], before the normative and stringent directions of the emerging "Analytic Philosophy of Mathematics" began to ravage the Anglo-Saxon world. Shaw explores *alternative mathematical spectra* with great insight: the geometrization of mathematics (chapter 3) *against* its classical arithmetization (chapter 2), the theories of operators, hypercomplexes and transformations (chapters 6-8) *against* logistics and deduction (chapters 5, 9), the analogical and dynamical forces in mathematics (chapters 4, 14-16), the centrality of form, invariants and functions (chapters 10-12) at the very *heart* of mathematical thought. Shaw offers direct comments on the *real Masters* of mathematics in the 19th century (Galois, Riemann, Kummer, Lie, Weierstrass, Klein, and, above all, Poincaré, his true intellectual mentor), on the French school in the Philosophy of Mathematics (Bergson, Brunschvicg, Winter, Milhaud, Boutroux), on Anglo-Saxon diagrammatical thinking (Hamilton, Sylvester, Peirce, Kempe), on the illustrious mathematicians of his time (Picard, Hadamard Russell, Hilbert). The panorama is entirely outstanding, going well beyond what many other texts will offer in the next hundred years, to really try to understand the "structure of mathematics" (transitions, spectra, dynamic forces) and its "central principles" (objects, form, invariance, functionality, ideality).

Albert Lautman's "Essay on Notions of Structure and Existence in Mathematics"[15], dedicated to the memory of his friend and mentor Jacques Herbrand, studies carefully the *structural* and *dynamical* conceptions of mathematics, interlacing the "life" of modern mathematics with a sweeping spectrum of dialectical actions: the local and the global (chapter 1); the intrinsic and the induced (chapter 2); the becoming and the finished — closely tied to the ascent and descent of understanding (chapter 3); essence and existence (chapter 4); mixtures (chapter 5); the singular and the regular (chapter 6). Lautman divides his thesis into two large parts ("Schemas of Structure" and "Schemas of Genesis") so as to emphasize one of his fundamental assertions regarding the mathematics of his epoch — that modern mathematics has a structural character (a prefiguration of the Bourbaki group, Lautman was close friends with Chevalley and Ehresmann) and consequently *mathematical creativity*

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[14] The lectures were given at the University of Illinois in 1915, to a "club of graduate students". I thank here Charles Alunni for pointing me to this truly extraordinary essay, almost entirely forgotten in the 20th century.

[15] His thesis for a Doctorate in Letters (philosophy), Paris: Sorbonne, 1937, *cf.* [Lautman 1935-42, 2011].

(the genesis of objects and concepts) is interlaced with the *structural decomposition* of many mathematical domains. Dialectical oppositions, with their partial saturations, and mixtures constructed to saturate structures, are linked to one another and to the underlying living processes of mathematical technique. For the first time in the history of modern mathematical philosophy, a philosopher conducts a *sustained, profound and sweeping survey of the groundbreaking mathematics of his time*. Confronting its technical aspects without ambiguity or circumlocution, and "dividing" it into basic concepts that he painstakingly explains to the reader, Lautman presents a strikingly rich landscape of the inventive currents of modern mathematics (his studies around Galois, Riemann, Poincaré, Hilbert may still be some of the best available introductions to their works). Thus, breaking with the usual *forms* of philosophical exposition, which used to (and, unfortunately, still do) keep the philosopher at a distance from *real* mathematics, Lautman opens an extraordinary breach in an attempt to seize upon the central problematics of mathematical creativity.

In his *Introducción al estilo matemático* [Lorenzo 1971], Javier de Lorenzo confronts some of the great figures of modern mathematics (Cauchy, Abel, Galois, Jacobi, Poincaré, Hilbert, the Bourbaki group, etc.) and argues that certain *fragments* of advanced mathematics — group theory, real analysis, and abstract geometries are his preferred examples — bring with them *distinct* ways of seeing, of intuition, of handling operations, and even distinct methods of deduction, in each of their conceptual, practical and formal contexts. De Lorenzo points out how mathematics "grows through contradistinction, dialectically and not organically", and thereby breaks with a traditional vision of mathematics, according to which it grows by accumulation and progress in a vertical ascent. He proposes instead a conceptual amplification of the discipline, in which new realms interlace with one another *horizontally*, without having to be situated one on top of the other. In *La matemática y el problema de su historia* [Lorenzo 1977], De Lorenzo postulates a radical historicity of doing mathematics. The references to advanced mathematics are classified in terms of three primary environments, within which, according to De Lorenzo, the major *ruptures and inversions* that gave rise to modern mathematics were forged: *the environment of 1827*, in which the program for the resolution of mathematical problems is inverted, setting out "from what seems most elusive in order to account for why [problems] can or cannot be resolved", and in which mathematics begins to feed on itself and its own *limitations*; *the environment of 1875*, in which the mathematical tasks of the previous half century are unified (groups, sets) or transfused from one register to another (geometrical methods converted into algebraic or axiomatic methods), generating the important constructions (Lie groups, point-set topology, algebraic geometry, etc.) that drove the development of mathematics at the outset of the twentieth century; *the environment of 1939*, in which the Bourbaki group fixed the orientation of contemporary mathematics around the notions of structure and morphism, inverted the focus of mathematical research, and moved toward a primordial search for relations between abstract structures (algebras, topologies, orders, etc.). In his studies, not only De Lorenzo prefigures the latter called "Philosophy of Mathematical Practice", but goes well beyond it in many aspects (contextual analysis, stylistic attention, horizontal translatability, historical acumen).

Ignoring the kind of efforts advanced by Shaw, Lautman, de Lorenzo, and by many other alternative "Philosophers of Mathematics" (*e.g.* Weyl, Mac Lane, Rota, Châtelet, etc.), the "Analytic Philosophy of Mathematics" has tried, unsuccessfully, to reduce Mathematics to Logic and Set Theory. Emphasizing, language, logic, and the fact that all mathematical concepts may be reconstructed inside the axiomatic architecture of sets, the reduction forgets that one thing is (1) to develop/create mathematics, and a very different one is (2) to present/represent those creations. Process (1) is done through images and structures which live in an intuitive *back-and-forth* between the discrete and the continuous, number and magnitude, algebra and topology, where a *mathematical* acumen is necessary. Process (2) is constructed, instead, through layers of language and axiomatic approximations, where a *logical* acumen is necessary. Often, the two processes have been disintegrated, showing no true understanding of "real mathematics" (meaning, for example, number theory, abstract algebra, geometry, topology, differential equations, functional analysis, etc.)

A true integration of processes (1) and (2) — the first one more synthetical/semantical, the second one more analytic/syntactical — becomes then a *minimum* condition to understand mathematics as a general form of thought. A simple way to ensure that minimum condition (a sort of *pendularity* between the analytical and the synthetical) is to understand *mathematics as a sheaf* (see *Figure 5*).

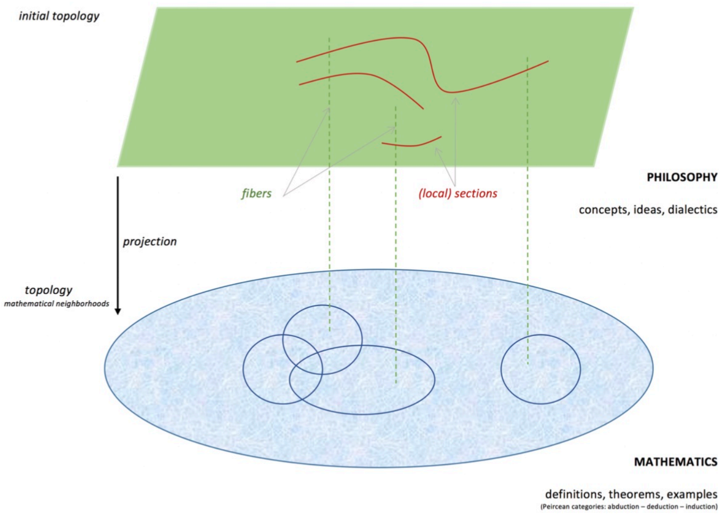


FIGURE 5. — Mathematical Thought as a Sheaf

In a bottom, *base* space, we place mathematics as it is usually presented — definitions, theorems, examples — emphasizing techniques, locality, necessity. Then, on an upper, *deployed* space, we place the fundamental ideas, images, intuitions that are projected and encrypted into the techniques of the bottom space. In this way, every technique (more akin to process (2) above) possesses over it a *fiber* of ideas (closer to process (1)). One of the main problems in order to understand mathematics as a true form of thought consists then to capture *sections* of ideas fostered by techniques, and to see if those *local sections* can be glued into *global* ones (determining thus some central forces of mathematical invention/creativity). This perspective forces at once a *natural shift* in the "ontology of mathematics": beyond objects *per se* (Absolute Ontology), or objects *per altri* (Transitory Ontology), objects are understood as photographs (fibers) of kinematic processes (sections) (Sheafification Ontology).

This simple model (S, for *sheaf* — topological locus of the study of obstructions and transfers between the local and the global) fails to take into account many other fundamental dimensions of mathematical thought: (a) the historical development of mathematics, (b) the stability of mathematics (layers of integral invariants — "archetypes" — beyond differential variations — "types"), (c) its irradiation to science and culture in general. Using some revolutionary ideas of Grothendieck (for a thorough introduction, see [Zalamea 2019]), one can expand the idea of a given, singular, sheaf, into a *multiplicity of sheaves* (T, for a *Grothendieck topos*) over a dynamic, evolving space (K, for a *Kripke model*) (see *Figure 6*). With these two expansions, we obtain a (TSK) model which handles well the requirements (a) and (b) above (historicity through K, stability through T).

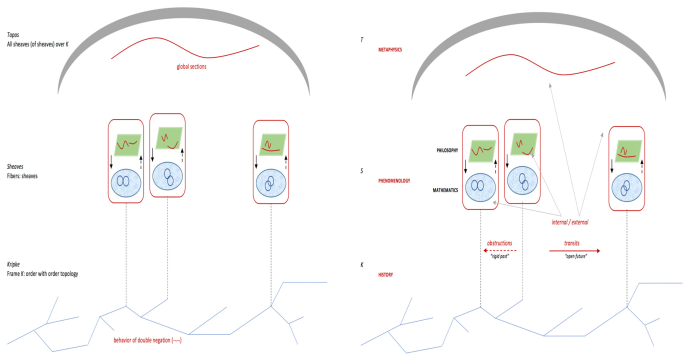
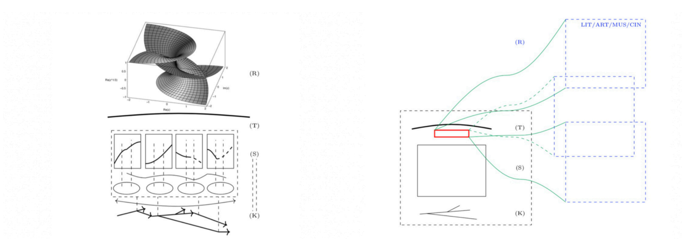


Figure 6. — Mathematics through a full (TSK) model

Finally, another expansion can integrate requirement (c), thanks to geometric ramifications (R, for a *Riemann surface*) (see *Figure 7*), where the many layers of culture (particularly around literature, art, and music) enter into a natural *dialogue* with the mathematical "archetypes" found at level (T). The result provides a rich, but simple, model RTSK (for a thorough discussion and many examples, see [Zalamea 2021]), where the complexity of mathematics is *not* reduced to one of its many dimensions.



Scheme of a model (RTSK). Case of a Riemann surface with three leaves *Cultural types (R) ramified over mathematical archetypes (T)*

Figure 7. — A model RTSK: mathematics (TSK) ramified through cultural types (R)

Working with the RTSK models, one can expect to develop some kind of sound *Mathematical Criticism* (expected result [Zalamea 2023]). In fact, the requirements exhibited in *Figures 2, 4*, and the desiderata listed at the end of *Section 1*, all are satisfied within the RTSK models. On one hand, the *full and faithful* functor *F* (*Figure 2*) is immediately realized by the many distinctions provided by the models: already at level of the "phenomenological" sheaf (S), *faithfulness* is obtained thanks to the good *topological* properties (local homeomorphism) of the projection between ideas (Mathematical Criticism) and techniques (Mathematics), and *fullness* comes handily from the very *set-theoretic* properties (surjectivity) of the projection. On another hand, the essential mediation, critique and inductive characteristics sought in a sound Mathematical Criticism (*Figure 4*), are obtained, in the mathematical realm, through the "metaphysical" level (T), and, in the cultural realm, through the "irradiation" level (R). Finally, characteristics (*i*)-(*iii*) for Mathematical Criticism listed at the end of *Section 1* are immediately incarnated in the RSTK models: *processes* (*i*) living at type (S) and archetype (T) levels, *modal/ideal critique of possibilia* (*ii*) living at invariance (T) and ramification (R) levels, *real entanglements* (*iii*) living at contextual, historical levels (K).

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