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Homage to Francis William Lawvere (1937–2023)

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When Jean-Pierre Marquis, after the death of Francis William Lawvere or simply Bill, as we affectionately called him, asked me to contribute some remarks in the mathematical/philosophical journal of which he is a co-editor, I accepted with enthusiasm, forgetting about the difficulties involved in such a task. In fact, I owe a great deal to Bill, both mathematically and philosophically and I can present only a small part of my debt.

As I mentioned in my Review Article "Topos Theory in Montréal in the 1970s : My Personal Involvement" [1], I had come to Montréal in 1967 with a PhD in Logic from Berkeley. As far as I remember, category theory was never mentioned in my years as a graduate student.

Much later I discovered that Lawvere had been in Berkeley from 1961 to 1963 as an informal student, following lectures by Alfred Tarski and Dana Scott before completing his PhD at Columbia in 1963 under the supervision of Samuel Eilenberg. Furthermore, Lawvere attended the 1963 International Symposium on "The Theory of Models" meeting in Berkeley, where he presented his functorial development of general algebra, and announced that quantifiers are characterized as adjoints to substitution. His article : "Algebraic Theories, Algebraic Categories, and Algebraic Functors" was published in the 1963 Proceedings of "The International Symposium At Berkeley" [2].

My first and most important mathematical contact in my new city was André Joyal. This contact was to prove determinant for a great part of my career. At the time of my arrival, Joyal was a student at Montréal University, working alone his way through Grothendieck's "Éléments de Géométrie Algébrique" [3] and several volumes of the "Séminaire de Géométrie algébrique du Bois Marie" [4]. I believed that it is where he learned category theory and toposes. I learned category theory mostly from him and he learned some model theory from me.

During these earlier years we studied Lawvere's and Benabou's articles whose aim was to do Universal Algebra categorically. And by 1970/1971, several algebraic structures such as groups and rings were "categorized". Joyal and I decided to extend their ideas to do first-order logic categorically, following the basic discovery of Lawvere that started categorical logic, namely that the categorical counterparts of existential and universal quantifiers are left and right adjoints to pull-backs (the counterpart of substitution). These are called images and dual-images respectively.

As Marquis and Reyes remarked in "The History of Categorical Logic (1963-1970)" [5], this was a key observation that convinced many mathematicians that this was the right analysis of quantifiers. They arise naturally as adjoints to an elementary operation, namely substitution, which appears as the basic operation of first order logic, contrary to the classical view which defines this operation by recursion, as a derived one.

Recall that if $f : X \longrightarrow Y$ is a function between sets X and Y and B is a subset of Y, then the pull-back of B along f is the subset of X defined by $f^*(B) = \{x \in X : f(x) \in B\}$. By considering B as a predicate of Y, say $B(y)$, the pull-back may be considered as the predicate of X obtained from $B(y)$, by substituting $f(x)$ for y. Lawvere's discovery was that the pull-back formation

$$f^* : P(Y) \longrightarrow P(X)$$

which is a functor between posets has a left and a right adjoints, namely the existential and the universal quantifiers

$$\exists_f \dashv f^* \dashv \forall_f$$

Indeed, by defining

$$\begin{cases} \exists_f(A) = \{y \in B : \exists_x(f(x) = y \text{ and } x \in A\} \\ \forall_f(A) = \{y \in B : \forall_x(f(x) = y \text{ implies } x \in A\} \end{cases}$$

it is easy to check that

$$\exists_f(A) \subseteq B \iff A \subseteq f^*(B)$$

which is the meaning that $\exists_f \dashv f^*$. Similarly, one can check that $f^* \dashv \forall_f$ namely

$$f^*B \subseteq A \iff B \subseteq \forall_f(A)$$

Lawvere's ideas motivated Volger to do first-order logic categorically in 1971 [6], but his attempt was not fully successful. Although we partly followed him, we realized that his mistake was to incorporate Ω, the sub-object classifier in the category associated to a theory, an object that does not belong to first-order logic. Once that we realized this, the road was open to finish the job. This was done by my student Jean Dionne in his Master thesis [7], not without solving some difficulties along the way.

Of all the results that Joyal and I obtained in the 1970s, the most important and most widely quoted was the existence of the classifying topos of an arbitrary coherent theory. It was discovered as an attempt to solve the main *contradiction* between Logic and Geometry in topos theory that Lawvere had formulated as the search of certain adjoint functors, see [8]. For a more complete exposition see [9].

In the meanwhile, Lawvere had obtained in 1969 the prestigious Killam professorship at Dalhousie University in Halifax, and was in that context allowed to invite collaborators in the subject of category theory. This meant that during 1969/1971, Dalhousie became the most important center of category theory. In particular, the notion of elementary topos was developed there.

However, in 1971, the group of collaborators at Dalhousie was dispersed; the university administration refused to renew Bill's contract due to his political activities. In fact, Bill had protested vigorously against the War Measures Act proclaimed by Pierre Elliot Trudeau, the Prime Minister of Canada, in October 1970.

The War Measures Act was the response to the kidnapings of a British diplomat, James Cross and a Québec Minister, Pierre Laporte by "Le Front de Libération du Québec" (FLQ), a militant Québec independence movement. Pierre Laporte was then killed the 17th of October and James Cross released on the 3rd of December after 60 days of captivity.

The War Measures Act gave the Government powers of arrest, detention and censorship for the third time in Canadian History and the first time in peacetime, hence, suspending civil liberties under the pretext of danger of terrorism.

My first encounter with Bill took place at the International Congress of Mathematicians in 1970 in Nice. I was in Nice by chance, waiting for a visa for Warsaw where I was supposed to study that year with Mostowski. This visa never arrived and I became stranded in Europe with my wife, Marie. At one point of our journey in Europe I met Adrian Mathias, whom I had known in Berkeley. He obtained an invitation for me to give some talks in Cambridge. This was the first time I gave talks on category theory.

The next time that I met Bill was in April 1973 in Montréal, where Bill spent that month, away from Perugia. At that moment in Québec, there was a lot of political activity from inside and outside influences. Small political movements invaded universities, labor unions and journals with discussions between Marxists-Leninists, Stalinists-Maoists, Mouvement Révolutionnaire des Étudiants du Québec, etc. From the beginning of the month discussions were suffused with dialectical materialism explained to us by Bill, with its emphasis on the universality of contradictions, main contradictions and so on. (See my paper Topos Theory in Montréal in the 1970's : My Personal Involvement [1]). This was an intense, informal gathering, not an organized, planned meeting.

One must not suppose that category theory evolved as a long, quiet, mathematical river. Bill had a very rich, complex and integrated mathematical, philosophical and political personality. His presence in a room was always felt, imposing but never arrogant or pretentious. He listened carefully to anybody who wanted to speak with him. He was very generous with his ideas and very serious in his beliefs that he defended with vigor.

The following year there was another meeting in Montréal which Bill attended. It took place in the summer 1974 and was organized by Shuichi Takahasi in the context of the "Séminaire de Mathématiques Supérieures de l'Université de Montréal". During this meeting, Bill carried out an incredible activity, giving talks, suggesting new approaches, encouraging some people and criticizing others. Some of these criticisms were carried out by means of "mass democracies" (a mass democracy is a collective accusation against a "reactionary" point of view of somebody). All these discussions and meetings left an unforgettable imprint on all the participants. The atmosphere was electric and sleep was very often missed.

Besides all this activity, Bill attended some meetings of the Communist Party of Canada (Marxist-Leninist). Once, he heard a long talk given by Hardial Baines, the First Secretary of the Party that made a great impression on him. Of course, all this activity took a toll and when Bill returned home to Buffalo, his wife Fatima told us that "she had to pick him up with a spoon". Lawvere had close relations with the Communist Party of Canada (Marxist-Leninist). I don't know whether he was a member, but the party wrote his eulogy.

This can be found in TML Monthly Newspaper of the Communist Party of Canada (Marxist-Leninist) No.1 January 2023.

Thanks mainly to the efforts of Bill, the theory of categories became very popular in Europe, North America, especially in Montréal, where a weekly seminar of categories lasted many years and also in Latin America, where I was invited to give several talks and seminars. In Bogota, people from the Universidad Nacional de Colombia and I organized a category meeting with Bill and Anders Kock as the special guests.

In 1975-1976, I obtained a sabbatical year which I spent mainly in Aarhus, Denmark. This was the beginning of a long collaboration with Anders Kock, the first student of Bill. From there on, I went to Aarhus so often during the summer, that once a professor seeing me in the corridor said : "Reyes!... It must be summer!"

During this and the following years Bill came often to Aarhus. Many people came to listen and some to work with him. During these stays I had the opportunity to deepen my understanding of his ideas that were not always easy to grasp. I often had the impression that his mind was meandering around the subject and I was surprised to see him arrive to the heart of the matter. He had an originality of thought that kept on surprising me.

A subject, among others, that shows Bill's originality and his way of working is "Synthetic Differential Geometry", also called "Smooth Infinitesimal Analysis".

It was the discovery of Lawvere that a "Grothendieck topos" may be viewed as a universe of "variable set" and that consequently set theoretic language can be interpreted directly in a topos. Therefore working from the topos built from schemes, rather than with the schemes themselves, one obtains a model for this generalized set theory with nilpotent infinitesimals or "Synthetic Differential Geometry".

To understand the motivation, we notice that "Differential Calculus", as we know it today was developed in the 19th Century by people like Weierstrass and others on a basis that was rigorous but far from being intuitive. This was not the way that the creators of the infinitesimal calculus in the 17th century (Newton and Leibniz among others) thought about these notions.

For instance, to define a tangent to a curve, they thought of the curve as the assemblage of an infinite number of infinitely small straight lines; or (what is the same thing) as a polygon with an infinite number of sides, each of an infinitely small length, which determines the curvature of the line by the angles they make with each other. This is rather complicated. Lawvere and Kock, on the other hand, turned the tables and considered an infinitesimal curve as a unique straight line. In symbols,

$$\forall f \in R^D \, \exists! \, (a,b) \in R \times R \; \forall h \in D \;\; (f(h) = bh + a)$$

where R is the ring of reals (not a field) and D the set of infinitesimals, namely, $D = \{x \in R \mid x^2 = 0\}$.

Indeed, the equation of a line is $y = bh + a$, where b is the slope and a is the initial value $f(0)$. Then the previous formula can be re-written as the Kock-Lawvere axiom or K-L axiom

$$\forall f \in R^D \, \exists! \, b \; \forall d \in D \;\; (f(d) = bd + f(0))$$

This says that every infinitesimal curve f is uniquely a straight line with slope b, *i.e.*

$$f(d) = bd + f(0)$$

Starting from here, Kock in his book "Synthetic Differential Geometry" [10] shows how the usual rules of differentiation can be derived.

The Kock-Lawvere axiom, however has an unexpected consequence : it is contradictory! as shown by Schanuel and Lavendhomme. Schanuel argument is given in Kock's book "Synthetic Differential Geometry" and is an example of the "law of the excluded middle" : define a function $g : D \longrightarrow R$ as follows :

$$g(d) = \begin{cases} 1 & \text{si } d \neq 0 \\ 0 & \text{si } d = 0 \end{cases}$$

If the Kock-Lawvere holds, $D = \{0\}$ is impossible. Hence, by using the law of the excluded middle, we may assume the existence of some non-zero $d_0 \in D$. By the Kock-Lawvere axiom,

$$\forall d \in D \; (g(d) = bd + g(0))$$

Substituting d_0 for d we obtain $1 = g(d_0) = bd_0 + 0$, which, when squared yields

$$1 = 0.$$

Lavendhomme in his book "Basic Concepts of Synthetic Differential Geometry" [11] proceeds by showing that under the hypothesis of the Kock-Lawvere axiom, it follows that R=0, the zero ring.

As Kock says, the moral of this story is that the Kock-Lawvere axiom is incompatible with the law of the excluded middle and hence one or the other must leave the scene.

I believe that this dilemma forces the law of excluded middle to leave the scene and forces us to weaken the logic, using intuitionistic or constructive logic rather than the classical one.

Lawvere also discovered that by considering smooth versions of the toposes occurring in algebraic geometry (toposes built from rings of smooth functions rather than polynomials) one obtains models for ordinary differential geometry. In these models infinitesimal structures of the kind used by Cartan, for instance, can be interpreted directly and, in this context, Cartan's arguments are literally valid. These ideas, dating from 1967, remained unpublished, and were taken up only in the mid-seventies. This resulted in two main lines of development. On the one hand, there was the purely axiomatic development of differential geometry with nilpotent infinitesimals, or "synthetic differential geometry" as developed by Kock [10] and Lavendhomme [11]. On the other hand, smooth toposes were constructed which show not only the consistency of the axiomatic approach but also provided a direct connection with the classical theory of manifolds. This second line was developed in the book "Models for Smooth Infinitesimal Analysis" written by Ieke Moerdijk and myself [12]. This approach avoids the three limitations of the category of manifolds discussed in the preface of our book. The basic ideas of this approach are due to Lawvere and can be seen to originate from the work of C. Ehresmann, A.Weil and A. Grothendieck . The aim is to construct categories of spaces, the so called "smooth toposes" which contain the category of manifolds (or more precisely, there is a full and faithful embedding of the category of C^∞-manifolds into each of these smooth toposes). Moreover, in each of these smooth toposes, *inverse limits* of spaces and *function spaces* can be adequately constructed, in particular, *infinitesimal spaces* like the ones needed in our interpretation of Cartan's arguments, e.g., the space D of first-order infinitesimals.

Lawvere had many followers and he was a source of problems at different levels of difficulty that he distributed freely among them. One could always find a problem that one could handle. One day, while in Denmark, Kock arrived at the office that was assigned to me, with a problem proposed by Bill that looked very interesting and maybe solvable : show that second order differential equations (ODE's) and many similar prolongation structures in a topos constitute another topos receiving an essential morphism from the first, provided that certain fractional exponents exist. We succeeded in proving this, partly following Bill's suggestions, and he announced our solution in Appendix 8 of "Outline of Synthetic Differential Geometry" [13] (see the list of "The collected works of F. W. Lawvere" : https ://github.com/mattearnshaw/lawvere/tree/master/pdfs).

Among the interests of Bill were also the history and teaching of mathematics.

In the list of "The collected works of F. W. Lawvere" there are several titles on the history of mathematics. One example : In "Grassmann's Dialectics and Category Theory", Bill says : "Looking more closely into Grassmann, Stephen Schanuel and I found numerous ways in which it could be justified to say that Grassmann was a pre-cursor of category theory. The general algebraic operations which he discussed have become the explicit object of a particular developed theory, and those general concepts, general operations, system of operations and systems of equation in invariant coordinate free form have been made into a part of category theory. More specifically, we find that in certain cases the famous distinction between analytic and synthetic operations can only be explained in terms of adjoint functors." (lawvere/pdfs/1996-grassmans-dialectics-and-category-theory.pdf p. 256)

Other examples can be found in the list of publications of Bill's works on the following authors :

Euler : "Euler's continuum Functorially vindicated"

Hegel : "Display of graphics and their applications, as exemplified by 2-categories and the Hegelian "taco".

Peano : "The relation of the so-called natural numbers to real mathematics".

Another aspect that I would like to mention is his passion for the teaching of Mathematics. In particular he wrote a book, and a collection of notes and papers.

The book, "Conceptual Mathematics (A First Introduction to Categories)", was written with his friend and colleague Stephen H. Schanuel, with the assistance of Emilio Faro and Fatima Fenaroli, Danilo Lawvere and the students of Mathematics 181 at the University at Buffalo. This book was translated into Spanish by Francisco Marmolejo with the help of Ivonne Pallares [14].

In this book, the authors said "Our goal is to explore the consequences of a new and fundamental insight about the nature of mathematics which has led to better methods for understanding and using mathematical concepts. While the insight and methods are simple, they are not as familiar as they should be ; they will require some efforts to master."

"Categorical Foundations of Set Theory and Logic" [15] is the title of a collection of notes and papers. In his foreword, Lawvere gives a preliminary description of the significance of each of the words in the title.

In particular, he explains "Foundations" as making explicit the essential general features, ingredients and operations of a science, its origins and its general laws of development. The purpose of making these explicit is to provide a guide to the learning, use, and further development of the science. A "pure" foundations which forgets this purpose and pursues a detached speculative "foundations" for its own sake is clearly a *non* foundations."

In 1985 I attended an intriguing lecture by John Macnamara, professor of Psychology at McGill University. He raised the question of giving a logical account of the phrase "Freddie is a dog" that children learn by the age of 18 months, claiming that logical theories were unable to account for it. I thought that he was mad : how can there be a problem with that ? But I found his talk so intriguing that I went to discuss with him at the end. This was the beginning of a collaboration between John Macnamara, Marie La Palme Reyes, Houman Zolfaghari and myself that lasted until John's death in 1996. I gave some talks about this subject in the category seminar in Montréal when Bill was present and, contrary to my expectations, he got interested in the subject and suggested several improvements. Furthermore, he accepted to participate in the Vancouver Meeting on "The logical foundations of Cognition" with a paper on the "Tools for the Advancement of Objective Logic, Closed categories and Toposes" that was published in the book "The logical foundations of cognition" [16]. His conclusion was that "Thus, I believe to have demonstrated the plausibility of my thesis that category theory will be a necessary tool in the construction of an adequately explicit science of knowing."

In 2007 I received an invitation to participate in the Boston Meeting on "Trends in the Mathematical Representation of Space : Philosophical and Historical Perspectives" to take place on November 31 and December 1, 2007. I was very surprised, since

I had never published anything on this subject, until I discovered that Bill was responsible for this invitation. In fact, I remembered that I had discussed with Bill the approach of R.K. Sachs and H. Wu to obtain Einstein field equations together with some simplifications that I obtained by using infinitesimals. My contribution, "A Derivation of Einstein's Vacuum Field Equations", was published in [17].

Looking back at my relation with Bill, I cannot but marvel at all what I learned and enjoyed through our conversations, our discussions and our common love of mathematics and philosophy. To accept that all this marvelous period has ended is very painful, however all the memories will live on in those who had the good fortune to know and appreciate him.

§ — **References.**

[1] Reyes, G. E. [2019] Topos Theory in Montréal in the 1970s : My Personal Involvement, Review Article, *History and Philosophy of Logic*, Taylor Francis Group.

[2] Lawvere, F. W. [1965] Algebraic Theories, Algebraic Categories, and Algebraic Functors, *The Theory of Models*, Proceedings of the 1963 International Symposium at Berkeley.

[3] Grothendieck, A. and J. Dieudonné [1960] *Éléments de Géométrie Algébrique*, Volume 1, Le Langage des Schémas, Paris : Publications mathématiques No 4, Institut des Hautes Études Scientifiques.

[4] Grothendieck, A. [1972] SGA 4 (Volumes 269, 270), Séminaire de Géométrie Algébrique de Bois Marie dirigé par M. Artin, A. Grothendieck, J. L. Verdier, Springer Verlag.

[5] Marquis, J. P. and G. E. Reyes [2010] The History of Categorical Logic 1963-1977, *Handbook of the history of Logic : Sets and Extensions in the Twentieth Century*, Volume editor : Akhihiro Kanamori, General editors : Dov M. Gabbay, Paul Thagard and John Woods, © Elsevier BV, All rights reserved.

[6] Volger, H. [1971] *Logical categories*, Thesis, Dahlousie University, Halifax, N.S., Unpublished manuscript.

[7] Dionne, J. [1973] *Des théories élémentaires aux catégories conceptuelles*, Thèse de maitrise, Université de Montréal.

[8] Reyes, G. E. [1974] From Sheaves to Logic, *Studies in Algebraic Logic*, Volume 9, Aubert Daigneault, editor,The Mathematical Association of America.

[9] Makkai, M. and G. E. Reyes [1977] *First Order Categorical Logic*, (LNM611), Springer-Verlag.

[10] Kock, A. [1981] *Synthetic Differential Geometry*, London Mathematical Society, Lecture Notes 51, Cambridge University Press.

[11] Lavendhomme, R. [1996] *Basic Concepts of Synthetic Differential Geometry*, Kluwer Academic Publishers.

[12] Moerdijk, I. and G. E. Reyes [1991] *Models for Smooth Infinitesimal Analysis*, Springer-Verlag.

[13] Lawvere, F. W. [1998] Outline of Synthetic Differential Geometry, Buffalo Geometry Seminar notes, February.

[14] Lawvere, F. W. and S. Schanuel [1991] *Conceptual Mathematics*, Buffalo Workshop Press.

[15] Lawvere, F. W. [1992] Categorical Foundations of Set Theory and Logic, Collection of notes and papers.

[16] Macnamara, J. and G. E. Reyes [1994] *The Logical Foundations of Cognition*, Edited by John Macnamara and Gonzalo E. Reyes, Oxford University Press.

[17] Reyes, G. E. [2011] A Derivation of Einstein's Vacuum Field Equations, Volume 53 of CRM Proceedings and Lecture Notes, Centre de Recherches Mathématiques de Montréal, *Models, Logics, and Higher-Dimensional Categories*, A Tribute to the work of Mihály Makkai, Edited by Bradd Hart, Thomas G. Kucera, Anand Pillay, Philip J. Scott, Robert A.G.Seely.

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