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Rota's variation on the eidetic identity

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Gian-Carlo Rota's conception of identity is based on his own understanding of Husserl's eidetic variation. This is evidenced by numerous publications.[(2)] Most of them refer explicitly to phenomenology, but others, more discreet, belong to another side of his activities which is not philosophical, but mathematical. Some of his colleagues, for reasons we will discuss later, hardly recognize such philosophical implications *in* a mathematical practice, or prefer to consider them as a mere ornament, to be left aside. Yet, the philosophical problem of identity is at the heart of any scientific identity and more precisely of mathematical identity.

To understand how Rota works and searches for mathematical identities, and what he means by "identity as eidos", I will begin by referring to when Rota became himself, that is, a *self-reflective* mathematician. I do not mean to say that working mathematicians are not reflective,[(3)] especially in modern times, but Rota is certainly reflective in a special way: phenomenologically reflective. Nor do I mean to say that he is the first: Husserl, as a mathematician, was certainly of this kind and others after him, like Hermann Weyl or Brouwer.

[(1)] This paper integrates part of a presentation given at the *Rota Memorial Conference*, 7/8 May 2019, organized by Albino Lanciani, Fabrizio Palombi, Marco Rigoli, held at the Università degli studi di Milano Dipartamento de Matematica Federigo Enriques, and completes analysis given in previous contributions (Lobo, 2017, and 2018).

[(2)] In his articles on Husserl in *Discrete Thoughts*, and in *Indiscrete Thoughts,* more particularly a chapter from an article written with Sokolowski and Sharp, entitled *Syntax, Semantics and the Problem of the Identity of Mathematical Items* (*Philosophy of Science,* 55, 1988, pp 376-386). A part is published in *Indiscrete Thoughts,* under a slightly different title *Syntax, Semantics and the Problem of the Identity of Mathematical Items* (Rota, 1997b, pp. 151-158). For this reason, we shall follow the first version (*i.e.* Rota, 1988).

[(3)] About reflectivity and the critical turn in mathematics, see H.-B. Sinaceur, Scientific philosophy and Philosophical Science, in *The Philosophers and Mathematics*, *Festschrift for Roshdi Rashed,* Ed. Hassan Tahiri, Springer, 2018, p. 56-63. And in the same volume, J.-J. Szczeciniarz, For a Continued Revival of Philosphy of Mathematics, especially p. 273-294.

But Rota is among the few who could so easily and explicitly articulate philosophically the series of motivated acts and decisions that constitute the ordinary work of mathematical thought, as well as the resources of mathematical virtuosity. Rota is undoubtedly such a virtuoso as well, but what sets him apart from many of his colleagues is that his scientific performances, at least after 1964, are accompanied by continuous and conscious critical reflection. This reflective turn in mathematics has been underlined, but this reflection is declared and explicitly phenomenological in Rota's case, that is referred to Husserl's transcendental and eidetic phenomenology, as it is evidenced by his philosophical lectures at the M. I. T.

I will begin with a brief presentation of his conversion to phenomenology. I will then present his philosophical account of identity as *eidos*, and of the phenomenological performance of eidetic variation, as well as his account of the ordinary treatment of identity in mathematics. Finally, I will offer a brief overview of some typical examples of such phenomenological awareness in his mathematical practice. I would like to talk about the notion of cryptomorphism and take as *eidê* some examples of mathematical identity.

§ 1. — The discovery of phenomenology by Rota.

The variations alluded to in the title do not refer to any change in Rota's views about identity as such, except perhaps the great change that was his conversion to phenomenology.

In a short article from 2000, *Ten Remarks on Husserl and Phenomenology*, he recounts how, after seven years of intense reading, Husserl's work suddenly began to become clear, to make sense, when for the "first time" he "managed to decipher [Husserl's] writing". This "happened one morning in March 1964". He was reading Husserl's *Ideas* in the back of the car, while his wife was driving.

> "As I was dejectedly rereading the conclusion of a long argument, *it suddenly made sense*. The *experience reminded me of one of those mixed multicolored engravings that hide an image, which you can only see if you stare at the engraving in a certain way*. My first reaction was 'So that's really it!', followed by 'Finally, I found the key'. I was young and brash. It took me a few more years to begin to understand. Every time I pick up one of Husserl's writings and

> start reading, I *get the same fuzzy feeling, and I've learned to wait for the hidden image to emerge*."(4)

As a statistical confirmation, let me add a personal note. What struck me, in reading this account for the first time, is that, while working on a PhD on Husserl under the direction of Derrida, and *independently* of any knowledge of Rota's work, I experienced the *same* feeling. And even more: the *same* analogy came to me, that between eidetic intuition based on a reading of Husserl's writings and the vision of stereoscopic image.(5) This analogy fully illustrates and captures how the phenomenological description and the eidetic variation work, and, consequently, it helps us to understand the concept of identity, here at stake.

Like the figure resulting from a tangle of colored spots thanks to a specific ocular adaptation (a certain *parallelization*, let us note), the *eidos* emerges from the arbitrarily changing aspects of ordinary perceptions, in an act of eidetic vision. But in a more general way, we discover retrospectively that any act of identification implies an ideation. We realize, says Rota, that "strictly speaking, we can never see the chair itself" but only "*examples of it*". And to avoid usual misunderstandings, we must insist on the fact that this "exemplary structure" or exemplarity, in Rota or Husserl's sense, is *not* an exemplification, and, consequently, it does *not* entail that perception should be infused by linguistic or conceptual structures. This eidetic exemplarity is prior to any expression or discourse, and underpins any act of identification, be it vague or acute, linguistic or prelinguistic, etc. Therefore: to perceive "*this*" chair, and recognize it *as* a chair, as well as to recognize the face, presupposes an eidetic apperception, which remains implicit and potential as long as we are not turning toward the specific or eidetic identity as such. But this holds also for higher objectifying activities: to *understand* a concept, to *discover* an unsuspected isomorphism between two distinct domains of mathematics, or to see and grasp an intentional essence from the reading and understanding of a series of propositions, of arguments, of a conceptual distinction, etc. — all these multiple and varied activities of identification that we call *knowledge*, which all tend towards a form of general objectivity, can be described in the same terms. Subsequently, as Rota concludes, "the

(4) *Ten remarks on Husserl and phenomenology*. O.K. Wiegand et al (eds.), *Phenomenology on Kant, German Idealism, Hermeneutics and Logic*, 89-97 2000 Kluwer Academic Publishers, p. 89. (Hereafter: Rota, 2000).

(5) C. Lobo, *Le phénoménologue et ses exemples*, 2000, Paris, Kimé, p. 16.

commonsense distinction between concrete and abstract collapses and becomes doubtful".[6] Moreover, before philosophy posed explicitly the problem of this eidetic identification, a kind of spontaneous eidetic variation is permanently and tacitly implemented at different scales and in different fields, under different objectifying attitudes, in the field of technics as Socrates remarked as well as in that of material eidetic, such as geometry, and the various branches of mechanics (statics, hydrostatics, etc.).[7]

It is true that an important difference remains in the phenomenological reflection, between the *performing of* an experience of identification and its thematic and phenomenological *description*. For this reason, I have compared the eidetic vision realized by phenomenology to the vision of a stereoscopic image and not to the mere focusing on an image or a perceptible object. The stereoscopic image presupposes a type of vision that is *neither* spontaneous nor natural. It is a vision or a focusing at two levels: a *vision at infinity*, which requires a parallax of our binocular vision, and at the same time a vision through the surface of the tangle of spots which begin to cover each other along a horizontal axis. After some time, a depth is produced, and the "phantom thing" appears. Similarly, a clear and explicit eidetic perception is quite difficult, and it takes a long time before it emerges, i.e., the time for the *adaptation of the view*, especially when it takes the form of a second order perception, a reflection on former eidetic formations. But how long did it take between the historical appearance of geometry and the first reflection on its hypotheses, with Riemann among others, which revealed the hidden metric or the specific topology at the basis of all Euclid's theorems? Or how long did it take for physics or astronomy to realize that the description of motions was meaningless if one did not assign coordinate systems and if one did not manage to explain the rules of their transformations? In addition, we must insist with Rota and Husserl, that the insight at stake in the eidetic intuition or ideation did not require any extraordinary faculty or mental power, but only a special capacity and a methodical exercise to push intentionality and attention in a particular direction. One cannot therefore accuse us here of resorting to a mystical faculty,

[6] G.-C. Rota, *The End of Objectivity*, Lectures at the M. I. T., by G.-C. Rota, 1974-1991, Second Preliminary edition. Drafted in Collaboration with Sean Murphy and Jeff Thompson, p. 180. (Hereafter, *End of Objectivity*).

[7] Axiological and practical attitudes, which belongs in the sphere of non-objectifying intentionality, would require different analysis.

as does the characteristic antiplatonism among philosophers. For I can imagine how Platonic this discourse on *intellectual vision* may seem and will provoke in many a form of skeptical reaction, if not outright hostility. Following Rota and Husserl, I will answer that Platonism as such is no objection. And I would go on to argue that since there are at least three of us who have experienced it, independently of each other, and reported on it in similar terms, that is an encouraging statistic for a start. Moreover, involuntarily, as many idioms in all the languages I know attest, we have a confused sense of such a vision. Rota expresses it as a spontaneous implementation of ideation.(8)

But again, these experiences as well as their spontaneous expression *are not* their methodological thematization; and even the manifestation of an act of ideation in a so-to-speak spontaneous phenomenological reflection is not methodologically sufficient, unless it is performed itself as an eidetic variation under a transcendental bracketing, that of transcendental *épochè.* Only then the identity becomes a *pole* of a multifarious thetic or positional activity, of perception, memory, imagination, symbolization... as well as for higher categorial objectifications as those at work in mathematics. This difficult but central point will help below to understand better the relation that Rota reveals on a formal perspective between syntax and semantics. From the point of view of phenomenology, which is that of the *a priori* of correlation(9) between the noesis and noema, — i.e., between the intending and the intended, the act and the content of act –, the theme is not only the *identity* on which one concentrates in each case, but the whole activity of thematization, the whole thetic or positional activity. We insist especially on the thetic or positional characters of acts and their correlative character of content (the posited, the theme or thesis, in the intentional and not only in the logical sense), because, as Husserl insists repeatedly, the spere of "positionality" constitutes the core of intentionality,(10) and explains subsequently in which deep and peculiar sense, that Husserl exposes in the lessons on *Passive Synthesis,* any cognitive activity is "logical", which does not mean discursive or linguistic.

(8) Rota et alii, 1992, p. 172-173.

(9) Husserl, 1950, see. §46, *Das universale Korrelationsapriori*, p. 161 passim.

(10) See for instance, Husserl, 1976, p. 333. For a brief overview of Husserl's reformation of the theory of "eidos" in its relation to the enlargement of the concept of "proposition", and "thesis", see m'y comment in « L'idée platonicienne d'eidos selon Husserl", op. cit. pp. 175-182, and more particularly p. 180.

In the epilogue to his conversion to phenomenology, Rota sent a message to Gödel praising Husserl as the greatest philosopher of all times. Gödel replied, adding that he was indeed the "true Kant", and the greatest philosopher since the time of Leibniz. And as Leibniz distinguished two great metaphysical labyrinths, that of *freedom* and that of the *continuum*, so Rota distinguishes two fundamental phenomenological problems: that of *time* and that of *identity*. Like the problem of the continuum, the problem of identity, although less vaunted and more technical, is surely the deeper one. "The problem of identity is equally important but less dramatic", says Rota, and at any rate much more difficult to "dramatize", although not deprive of some dramatic aspects.

§ 2. — Philosophical account of identity as eidos, and the phenomenological performance of eidetic variation.

The problem of identity, in its most schematic form, asks *how a* thing, let us call it x, can occur in different instances, be presented in different forms or modes. This implies conversely that something appears as being *an* x or *that* x, insofar as it is an *instance* of x, and therefore appears *as* something that constitutes its identity, and that goes beyond its factual being. This concentrates the essential features of the type of *vision* that we mentioned initially, by comparison with the perception of the stereoscopic image.

This form of identity is the *eidos*, the similitude that flows from one form to the other, while the transitory carrier can be called either *an* instance of the *eidos* or more generally its facticity. The things that appear in the naive attitude do not appear for themselves, but *for* (or as) something else. We only come to this understanding of the *what* because we seek to keep an eye always on the *how*. To observe *how* this reference is obtained requires us to depart, so to speak, from our ordinary attitude. This is the shift to the phenomenological transcendental attitude, through a universal bracketing. On this basis alone, can start the analysis of the structures of acts, of their composition and foundation (*Fundierung*), since higher order cognitive activities take the form of an articulated series of founded acts. Following Rota's terminology, the founding act gives the support or *facticity* to the higher and founded act, which is *in function* (*fungierend*, in Husserl's terminology); and as I say the former functionality of the founding act is neutralized, disactivated, put outer function. Or in other terms, there is no irreducible or absolute facticity. Once clearly distinguished facticity and function, it is essential at the same time to keep in mind the relativity of *function* to *facticity*, and vice versa, in order to understand why and how the eidetic or specific identity is itself contextual.

> "To say that two marbles are identical is to say that the two marbles have identical *eidos*, not that they have identical facticities. The identity is the identity of the *eidos*. They can be identical because both can have the same *eidos*. In other words, there is confusion about identity only if we confuse facticity with function, as happens with physical reductionism. If we stay away from reductionism, there is no confusion about identity, and thus there is no confusion about how a decontextualized *eidos* can be contextually particular 'this'."[11]

Now, all identity is given through a multitude of modes of giving, and exploring these modes is crucial to avoid any form of reductionism. But because we are usually tended toward the object whatever it may be, in a process of identification, we focus mostly on the *what* and neglect the *how*. As a result, since the *what* is contextual, we have a confused idea of the *what* itself, and a rigid and decontextualized notion of identity. Reductionism is the typical reaction, the compulsion towards such a rigid identification, expressing anxiety about the fact that *all* identities are *context dependent*, functionally based on multiple changing facticities and therefore never completely fixed. Or, what amounts to the same thing, to be concerned with the how, is to *look at* the reference *in its context*, *as context dependent*, *in its "fringe" of given modes of being*. As Rota insists:

> "One of the characteristics of phenomenological description [...] is that, in the description of contextual phenomena, we give equal importance to the 'fringe'. In fact, one of the great differences between a purely objectivist description and a phenomenological description is that, in the latter, all kinds of marginal phenomena are given equal importance."[12]

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(11) *End of Objectivity*, p. 184.
(12) *End of Objectivity*, p. 75.

Because he neglects the how, the objectivist — or reductionist — has neither an idea of the "things themselves", nor even a clear idea of the *what* itself[13]. Any kind of objectivist or positivist is reductionist by restricting his field of action to what he has first posited as the real object, forgetting the positional activity, and the way in which this position presupposes the prior position of the world. Against this restriction, the bracketing of the phenomenological description enlarges the field to the widest horizon, and at each stage of the investigation includes in its perspective the *how* of *what* is at stake. For this reason, Rota diagnoses in the very process of objectivation that leads to an objectivist attitude a resistance to this widening, an inner and unconscious protection against the phenomenological bracketing.

There is an affective and axiological side to this process too,[14] which Rota calls "reductionist anxiety". This anxiety is at the background of any epistemological attitude affirming that any "thing" that cannot be reduced to a region (psychic, physical) supposedly already and rigidly identified is considered as nothing. This anxiety reaches its paroxysm when one is confronted with identity *as such*, when one is about to recognize the identity of a thing, of everything that escapes us and that does not fit into the framework of a ready-made and fixed ontology. This is exactly what happens when we are first exposed to phenomenology, which focuses on the eidetic intentional structure of each region of being, and that, as we become familiar with this kind of approach, a feeling of instability sets in.[15] Reductionist anxiety stems from it.

As we have said, there is a huge difference between the implicit, tacit and spontaneous eidetic variation, with the specific parentheses that accompany it, and the explicit, and methodic variation. But does this mean that all knowledge is implicitly phenomenological knowledge? If the *what* of knowledge is the *eidos*. What about the *how*? Rota answers:

> "We know in many ways. At first sight, it seems that the scientist knows by examining tables of data and by observing scientific laws, that mathematicians know by giving proofs, and that the everyday man knows by simple calculation. [...] What is primordial in all these phenomena that can be singled out as the philosophical

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(13) Against reductionism, identity, *End of Objectivity*, p. 34-35, p. 43

(14) On axiological predicates, see *End of Objectivity*, p. 5, p. 10-11.

(15) *End of Objectivity*, p. 60.

meaning of knowledge? Each of these ways of knowing is a *process of bracketing by eidetic variation*. In other words, phenomenologically, we assert that even through various instances, there are many ways of knowing, primordially there is only one, which is the bracketing of an *eidos* through eidetic variation."(16)

In the standard presentation of the phenomenological method, bracketing is the transcendental reduction and *epochè*, which transforms the world into a world-theme, where the naïve belief in its existence is suspended, and, as a consequence and correlatively, the *world-thesis* becomes a theme for a phenomenological description. This is another aspect of Rota's understanding and appropriation of Husserl's phenomenology with which I fully agree, especially concerning the inter-implication between eidetic variation and *epochè*. Rota's thesis is that "every phenomenal vision" implies an eidetic vision, and an eidetic variation.(17) And one of the philosophical goals of phenomenology is precisely to describe this *eidetic vision* in its essential characteristics. Rota enumerates three of them: 1) "we look to familiarize ourselves" with the issues, to disclose an intentional meaning. 2) "We see" insofar as we have a "project", i.e., an intention with its horizon-structure. 3) "We look" insofar as we look beyond what we are looking at. Such a "beyond"(18) motivates the deceptive analogy between sensual appearances and signs (that before Galileo was already suggested by Democritus).(19)

§ 3. — Rota's explanation of the ordinary treatment of identity in mathematics.

Among Rota's most surprising statements about mathematical identities are certainly the following two, which seem to contradict each other to some extent.

S1: Mathematical identities are ultimately (abstract) individuals.

S2: They are given, through a series of profiles and under many aspects, as any real and worldly individual.

(16) *End of Objectivity*, p. 182.

(17) *End of Objectivity*, p. 174.

(18) *Ibid*.

(19) Husserl's critique of this analogy and that of the distinction between primary and secondary qualities, in *Ideas* I, § 40, (see Husserl, 1976, 82 passim), has triggered many misunderstandings on the relation of phenomenology with modern sciences, and consequently on the diagnose of crisis of European sciences (Husserl, 1950) and the role that transcendental phenomenology should play in it.

S1. The mathematical objects (idealities) are (abstract) individuals. Contrary to the (concrete) individual objects, they are not identities; but the *eidos* is confused with it. The real line is not an instance of anything. "Most mathematical objects have [the] property that there is only one. For example, there is only one real line, even though it has many properties", and can be formalized in different axiomatic perspectives. "No mathematician says there are many [real lines]- it is the same line, loaded with the properties of the real line."(20) The same analysis is repeated and developed elsewhere:

> "The objects of mathematics, objects such as the real line, the triangle, sets, and natural numbers, share the property of retaining their identity while receiving *axiomatic presentations* that can vary radically. Mathematicians have axiomatized the real line as a one-dimensional continuum, as a complete Archimedean ordered field, as a real closed field, or as a system of binary decimals on which arithmetic operations are performed in a certain way. Each of these axiomatizations is *tacitly understood* by mathematicians as an axiomatization of the *same real line*. In other words, the mathematical object thus axiomatized is presumed to be the same in each case, and this identity is not questioned. In this paper, we wish to analyze the logical conditions that make it possible for *the same mathematical object to be designated by a variety of axiomatic presentations*." (21)

S2. However, mathematical objects are given through different (abstract) profiles, through different (abstract) perspectives. Like any scientific object: say the earth, the universe, but also matter, motion, life, DNA, etc., the real line, the number three etc. are the poles of endless processes of objectification (determination, identification, clarification, etc.). The similarity or identity in both cases is given by a series of profiles that provide the facticity for endless presentations of the *eidos*. These profiles are not constituent parts of the thing itself, nor are they even aspects or faces of it.

More, as any facticity can become the object at the core or the process of identification, it can also become the pole given through profiles. For instance, there is only one north face from a determinate

(20) *End of Objectivity*, p. 181
(21) Rota, 1988, p. 376.

perspective of the Duomo (perspective including determinate distance, height, focus etc.), and an infinity of profiles of it, which stems from the variations in the context (changing light, changing color, etc.). Of course, this "face" becomes itself, in this case, a context-dependent ideality. Similarly, the real line can be accounted for from many different angles. But since the real line is not a being existing in the world or in *a* world, what can be the profiles and what about its faces or aspects? Rota's answer is surprising but straight to the point. The profiles of the real line are *formal presentations*, and more precisely *syntactic or semantic presentations*. This implies that axiomatic systems are not a theoretical goal *in themselves,* but mere formal presentations, i.e., approaches, in an endless process, to mathematical eidetic identities.

Rota's understanding of Husserl's analyses of mathematical idealities is very acute and profound, and it departs from the current philosophical discussions on "mathematical existence", because, insofar as the individual identity merges with the *eidos*, its existence is equivalent to its possibility.(22) Rota thus takes the side of a "synthetic" (which does not mean necessarily constructive) philosophy of mathematics, a philosophy attentive to the real work of mathematicians:

> "By saying this, we recognize the real practice of mathematics, and we do not plead for Platonism: we are not in the least concerned by the *problem of the existence of mathematical objects*, any more than the grammarian is concerned by the problem of the existence of the verb or the adjective. Our argument aims at showing that the problem of the identity of mathematical objects is not solved, but rather exacerbated, by a doctrinaire appeal to axiomatics."(23)

In passing, this would shed new light on the problem of the *imaginary* in mathematics which was at the center of Husserl's philosophical reflections on the concept of number. An impossible concept within the frame of a given theory, such as $\sqrt{-1}$ can be syntactically required, for a coherent axiomatic presentation of an eidetic identity whatever it maybe(24).

(22)On "*Wesenhaftigkeit*" (essentiality) which is tantamount to "*Geltung*" (validity), see Husserl's *Prolegomena*. (Husserl, 1975, 241)

(23)Rota, 1988, p. 382.

(24)See Husserl's solution to the problem of the introduction of imaginary numbers in, *Philosophie der Arithmetik, mit Ergänzenden Texten* (*1890-1901*), Hsg. Lothar

This entails a series of consequences**.**

1. The mathematical *ideation* is given in a perspective where an axiomatic system thus becomes a sketch or a formal profile of an object. This means that the eidetic vision is structured like the empirical vision. In both cases, the identification is a never-ending process, never completed. It never reaches a complete and ultimate certainty and evidence. Shocking as it may seem, any axiomatic presentation of a mathematical identity must be consistent but can never be complete: the mathematician wants to find out what else the real line may be[(25)] — *and there is always something else to find*. The "real line, or any other mathematical element, is never completely given in an axiomatic system", or in a finite series of axiomatic systems.

> "It is impossible to state all the axiom systems for a given theory. Think for example of the theory of groups. One has to introduce several axiomatizations of the group concept, which may even use ternary or more general operations. When a mathematician designs a new system of axioms for a known theory, for example group theory, he is already guided by an already intuited notion of group."[(26)]

2. Although mathematical approximation never gives rise to any *rectification* (refutation, "falsification", etc.) as in the empirical sciences or in ordinary knowledge, there is nevertheless an open horizon of possible determinations anticipatively determined, conjectured, or demonstrated. The limit concept of the set of all axiomatic presentations of the same object, let us say the "real line, can neither be explained nor foreseen in an exhaustive way". Consequently, the idea of a "totality of possible axiom systems for the real line cannot be exhaustively stated or anticipated". The ideal of completeness is thus defeated: "Every mathematical object allows for an open sequence of presentations by ever-new systems of axioms; that is, the successive development of ever-new systems of axioms for what is perceived as one and the same object."[(27)]

Eley, Husserliana 12, M. Nijhoff, 1970, pp. 441 and 444. Two conditions must be satisfied: 1. The imaginary number must be axiomatically consistently defined; 2. All propositions within this axiomatic system must be decidable. In order to avoid introducing an extrinsic completeness axiom, the axiomatic system must allow an enlargement, that is, the introduction of a new axiom, if it stumbles upon an undecidable proposition.

(25) Rota, 1988, p. 383.

(26) Rota, 1988, p. 381.

(27) Rota, 1988, p. 381.

We find a similar analysis in *The End of Objectivity*, with a critique of a narrow conception of formal logic, the "myth of a standard logic" which would answer the requirement of the absolutist thinker, as if objects, including so-called abstract objects, say mathematical objects, did not depend on context. But formalization remains necessary and plays itself a heuristic role, by making evident the gap between the actual practice of mathematics and its formal account. Axiomatic systems reveal aspects of the mathematical object ($= X$) that were not manifest *as such*. This is the case of the formalization of the notion of group to classify an enormous amount of pre-existing special groups, which led to an *a posteriori* discovery: "the most significant examples of groups were discovered after the abstract concept of group had been isolated by an axiomatization. Even if mathematicians consider explicitly classifying all models of groups, for example all finite groups, the practice of mathematics nevertheless belies such explicit programs. Examples of mathematical structures for which all models are found are rare (the most successful in recent memory being Elie Cartan's classification of simple Lie groups)."(28)

3. As a corollary, formal logic must itself be extended to take account of the operations of *searching* for an axiomatic system themselves and of the deliberation leading to prefer one axiom system over the other. The general motive being that a certain axiomatic system gives a *better presentation* of a new property of the mathematical object under consideration: "each of these systems is supposed to reveal new characteristics of the mathematical object."(29) Thus every formal approach is itself dependent on a *seeing*, on an informal anticipation of an eidetic content, i.e., a "*pre-axiomatic grasp*" of the mathematical identity that functions as a thread. Once a formal presentation of an aspect of the object is grasped and trivialized, once we have familiarized ourselves with the object through a certain perspective, the intuition can be put aside. The same thing happens with the grammatical rules of a language, once we understand those rules and become a fluent user of that language, the rule is set aside, or in the background of our attention and concern. Yet, without these rules, our speech would lose its meaning, just as without this "guiding intuition, even if not verbalized"(30), any formal system would lose its meaning and could not be motivated and designed. This happens

(28) Rota, 1988, p. 383.
(29) Rota, 1988, pp. 383-384.
(30) Rota, 1988, p. 384.

not only with seemingly geometric objects, but also with structural and highly algebraized objects, such as a group.

> "We will call 'pre-axiomatic grasp' the understanding of the notion of group that has been freed from the necessity of choosing a system of axioms. Since there is no single group, but groups are of incredible variety, such a pre-axiomatic understanding of the notion of group cannot be attributed to familiarity with a single group. One could argue that such an understanding is derived from familiarity with more than one group and their 'common' properties, but this would be begging the question, since *familiarity with more than one group presupposes an unstated understanding of the concept of group that allows one to recognize multiple instances as instances of the same general mathematical structure*."(31)

As underdetermined or blurred this "eidetic identity" may appear, it is presupposed by each instance that fall under consideration.

4. Since "a complete description of the logical role of a mathematical object is beyond the reach of the axiomatic method as it is understood today", the phenomenological description calls for a new logic, extended to cognitive operations, such as logical individuation, modalization, foundation, or relations such as context dependence, etc.(32) Identification as the presentation of an *eidos* through formal profiles is one of these operations. This position is common to Rota, Husserl and Stanley Ulam, who was convinced that "logic formalizes very little of the processes by which we think" and that "the time has come to *enrich formal logic with other fundamental notions* [emphasis added]. What do you see when you see? You see an object *as* a key, you see a man in a car *as* a passenger, you see some sheets of paper *as* a book. *It is the word 'as' that has to be mathematically formalized, just as the connectors 'and', 'or', 'implies' and 'not' have already been accepted in formal logic*. Until you do that, you won't get very far with [the] A.I. problem."(33)

(31) Rota, 1988, pp. 381-382. Emphasis mine.

(32) On *equiprimordial* modes, see *End of Objectivity*, p. 13, p. 15; specifically on modalities, excluded by standard formal logic, *Ibid.*, p. 19; on context dependence, see 8ff. and on *Fundierung*, *Ibid*, see p. 41.

(33) Barriers of Meaning, in *Indiscrete Thoughts*, (Rota, 1997b, pp. 58-59).

We find the same demand for a formal account of a larger number of operations that come into play in any cognitive process, including affective stances, which would in any case require and presuppose a careful phenomenological investigation. But the question remains: "Can a rigorous formal analysis supplant what was a psychological description of intelligently performed operations?"[34] For example, are operations such as *comparison*, *theory formation*, *example citation*, and *reference* etc. formally and eventually algorithmically representable?

5. This implies a new relation of the logical problem between syntax and semantics, between an axiomatic system and its interpretations. Models are classes of mathematical objects satisfying certain syntactic rules, where each model includes an additional structure. The "paradigmatic example" is given by set algebra[35], which has provided the ideal example of a successful formalization of a mathematical theory, (1) where a syntax is specified with intuitive inference rules, together with a complete description of all models for such a syntax, in this case sets, (2) "and, what is more, a table of semantic concepts and syntactic concepts, together with a clear translation algorithm between the two. Such an algorithm may, for the purposes of this discussion, be called a 'completeness theorem'."[36]

6. The conception of formalization itself must be modified accordingly, to *preserve* the structural incompleteness of the mathematical horizon. Informally, this amounts to relating any identity to its context, i.e., to the "bundle of intended models" surrounded by "an unclassifiable variety of unexpected, unwanted, 'non-standard' models" and yet sought after. Semi-formally, Rota expresses this situation as follows:

> "If an axiomatic system A is used to describe a model X, and another axiomatic system B is also used to describe the same model, the contribution of system B is desirable because it shows *what* X can be *other* than what it is shown to be by A. The contribution of B is not desirable simply because it confirms what we know about X by A,

[34] Rota, 1988, p. 385.

[35] "Multisets can be added and multiplied; however, a characterization by algebraic operations of the family of multi sets of a set S — an analog of what Boolean algebra is for sets — is not known at present. The problem is of more than academic interest. There is a deep duality between the algebra of sets and the algebra of multisets that a syntactical description may well elucidate. [19, Chap. XII. Syntax, semantics, and the problem of identity of mathematical items". Rota, 1988, p. 154-155]

[36] Rota, 1988, p. 379.

but because it reveals something else that is true about X. It allows X to be presented more fully and to be re-identified in another context."(37)

7. This analysis and similar better explain Rota's and Husserl's common project of a reform of logic and, in Rota's case, the reformed model theory. This reform entails a deconstruction of the absolutist and objectivist norm of formalization and of the usual logical understanding of the relation between syntax and semantics. Of course, logical algebra from Boole to Halmos and beyond provides the standard for all semantics and syntax, but, as Rota notes, the discovery of a formal hidden analogy (a "cryptomorphism") usually comes afterwards, only once the mathematician becomes familiar with two models:

> "Although rigorous criteria have been stated for the *equivalence of two axiomatic descriptions of the same* object (e.g., in first-order logic), these criteria are often expressed as if the mathematician were suddenly faced with the problem of verifying that two previously given axiom systems are in fact *cryptomorphic*. This verification is sometimes requested, but rarely. In *reality, the mathematician is normally confronted with two axiom systems when he is already fully aware of the identity of their models*." (38)

In order to understand this ternary relation between syntax, semantics and identity as eidos, let us come to the analysis of some examples.

§ 4. — Overview of some typical examples of such practice in Rota's mathematical work.

1. *The Reynolds operator*

While working on the Reynolds operator, Rota realized that the averaging operator used in many fields was formally analogous to the Navier-Stokes equations, when properly simplified. This opened a new field of investigation and repeated attempts to connect parts of logical quantization and probability measures on sigma-algebras, where the Reynolds operator appears as a mathematical identity.

(37) Rota, 1988, p. 383.
(38) *Ibid.*

Pushing forward Halmos's program of algebraic logic, Rota mentions, as possible application and as "the most promising outcome", the *translation* of "the notion of quantifier on a Boolean algebra" into that of "linear averaging operator": "in this way, problems in first order logic can be translated into problems about commuting sets of averaging operators on commutative rings".(39) The conclusion of the paper we are following here relates this translation of predicate logic to that of probability, which gives the starting point of a contribution with D. P. Ellerman (*A Measure Theoretic Approach to Logical Quantification*) (Rota, 1978), including the cryptomorphic bridge between probability measure and quantification. In its presentation from 1973, the analogy between probability and predicate logic is mediated through the following steps: assignment of a probability measure m to the canonical idempotents; definition of a lattice space norm on the valuation ring by using a linear functional, complementation of the resulting normed linear space which can be seen as the space of all integrable functions, representation of the averaging operator as a conditional expectation operator, which, once relaxed from the restrictive condition that "every element of the range be finite-valued", can apply universally.(40) Rota Remarks: "An averaging operator on a valuation ring $V(L)$ is a linear operator A such that: (1) $Au = u$, $Az = z$. (2) $A(fAg) = Af\,Ag$. (3) If [the function] f is in the monotonic cone, so is Af. Sometimes these operators go by the name of Reynolds operators. In probability, they are called *conditional expectations*." We have here, as it has been noticed by Jean Dhombres, a perfect example of Rota's synoptic ability and his quasi-polymath profile.

> "Rota was a master at perceiving connections among disparate subjects and in this instance, he took a combinatorial point of view. In particular, he used canonical idempotents instead of characteristic functions, borrowing the idea from Louis Solomon's work on Burnside algebras [...]. But something else was occurring Rota *was no longer analyzing the structure of averaging operators, but, constructing averaging operators to fit the application. For the study of logic, he constructed specific 'averaging' operators on valuation rings, which simulated the properties of existential quantifier*. In this way, 'Problems of first order logic, such

(39) Rota, 1973a, p. 576.
(40) Rota, 1973a, p. 622.

> as the decision problem, can be shown to be equivalent to algebraic problems for valuation rings with averaging operators.' *Unfortunately, it proved quite difficult to carry through this program rigorously, as one can see in the paper* [...]*, written with David Ellerman*"[41].

Following Birkhoff example, who always kept an interest in logic, Rota inverts what would be the current perspective for mathematicians. And the eventual disciple and commentator's surprise and deception echoes the reaction of Rota's teachers in mathematics, who did not understand his interest in formal logic, and could even less guess his philosophical motivations. As to the latter, they did not stem from the deceiving perspective of some philosophers misled by the "fake philosophical terminology of mathematical logic" "into believing" that it "deals with the truth in the philosophical sense". According to Rota's terminology, this variety of believers constitutes precisely what he names philosophers under "a bad influence of mathematics", which relies on a *misunderstanding* concerning mathematics as well as philosophy and falls into that typical artificial philosophizing whose symptoms, in certain schools of philosophy, are: "casual and self-satisfied symbol-dropping of mathematical logic" and contempt for history and evidence (or intuition), especially in the Husserlian sense of the term.[42] Abandoning the belief that mathematical logic had something to do with truth and scientific truth was the due price that logic had to pay in order to become "a successful and respected branch of mathematics".[43] By giving up any philosophical pretention means that logic can be treated as any other branch of mathematics, and that we should not inject into it any philosophical meaning. Yet that does not mean we dismissed any research concerning a philosophical logic. On the contrary, we should keep logical questions opened. More: since mathematical logic has either nothing to do with the way we think, (as some logicians think) or, in the better case, as already stated, "formalizes only very few of the processes by which we think", *philosophic logical* investigations should remain at the same time an open field as well as an open source for eventual enrichment of formal logic.[44] Whatever the success or failure of Rota's attempt to reconstruct algebraically

(41) Dhombres, 2003, p. 163.

(42) Rota, 1991, 134 & 135, respectively.

(43) Rota, 1997b, 97. Rota, 1991a, 133-136.

(44) Rota, 1997b, p. 92-93.

the quantifiers, and in order to decide if he succeed or failed, it is worthwhile seizing precisely what are his logical aims. While doing so, we gain a better understanding of the strange transference of an operator from a mathematical field (constructed in the context of fluid mechanics) to logic (predicate calculus), via algebra. I mean Reynold's operator.

Although he recognizes a wider scope to Rota's exploitation of "the idea of averaging operators"(45), Dhombres considers that the main mathematical goal that Rota had in mind in using the Reynolds equation (a generalized form of the Navier-Stoke equations)(46) was a broadening of the field of applications (models) and a unification of the theory of measurement.

The Reynolds operator is an averaging operator, i.e., a heuristic way to simplify the Navier-Stokes equations which are fundamental for hydrodynamics (description of gas and fluid flows, etc.) and beyond. Their form is quite complex, combining three quantities of material elements (mass, momentum, energy) whose density is measured; the velocity field and the energy density per unit mass. From the beginning, Rota is interested in representations (i.e., models) of the Reynolds operator, in ergodic theory, in spectral theory and in probability theory. The simplified form is written $\mathrm{R}(fg) = RfRg + \mathrm{R}\{(f - Rf)(g - Rg)\}$. (NB. The mean form is: $A[f(Ag)] = (Af)(Ag)$, with f, g two functions belonging to a commutative algebra). The transition from the Navier-Stoke operator to the Reynold's operator is smoothed, so to speak, by considering the functions of the Navier-Stokes equations as *random functions.* The Reynolds operator is thus seen as a *conditional expectation operator* of the form: $E^{\Sigma'}(uE^{\Sigma'}v) = E^{\Sigma'}uE^{\Sigma'}v$. (47)

Dhombres comments:

> "In Rota's mind, the mathematical task was to understand the proper role of the identity [equation], but *not in an axiomatic or formal way. He thought that studying a variety of situations in which Reynolds operators played a*

(45) "In 1973, Rota exploited the idea of operator averaging in the radically different context of valuation rings of a distributive lattice in [50] [i.e., here (Rota, 1973a)]. This paper is fascinating because, as Rota writes, 'the method of presentation is deliberately informal and discursive. Much of the reasoning is by analogy, but not in the structuralist or Bourbakist sense, and many of the results are merely heuristic'. The objective is grandiose, nothing less than the 'linearization of logic'." (Dhombres, 2003, 163).

(46) *For* more details, see: Rota, 2003: p 137-139.

(47) Rota, 1960, 138.

> *role in the real (or phenomenological) world would provide a better understanding of the meaning of the Reynolds identity.* Although Rota believed that a mathematician should be inspired by the physical world to do pure mathematics, he disagreed with the view of those mathematicians who insisted on the primacy of problems arising from technology that had to be solved, whether or not they led to pure mathematical results."(48)

It is true that Rota did not *only* understand applications in an empirical sense but *also* had in view formal models (and as we have seen, vice versa). Significantly (recalling Rota's words as supervisor of his PhD thesis, in 1970), Dhombres reports "how he insisted on concrete realizations of such operators on different function algebras *and* how he focused on how the Reynolds identity and related identities could simplify known results"(49) ; and even how the Reynolds operator gradually disappeared from his writings, after 1974, since it might seem too complicated or too rich. But, at the same time, even before that, from 1964 on, Rota no longer claimed to use it as a means of producing a unified theory of measurement and had to take it as a simple game that could generate "calculations that could be used or illustrated by examples in functional analysis". The main reason given by Dhombres was that there were alternative and equally effective "tools" for the same purpose, "such as derivations and Baxter operators"(50). From such a change, we should assume either an excessively adaptive attitude or an inconsistency in Rota's position, moving from one project to another.

His position in and vis-à-vis the history of mathematics should be, in both senses of the term, discrete (discontinuous and anecdotal).

> "For Rota, *progress in mathematics could sometimes be made by recognizing a pattern or by working by analogy*. Reynolds' operators [sic], which had been a *kind of exercise* for the mind, remained *for him a source of inspiration*. This suggests that other authors, even if they do not work specifically on Reynolds operators, might also be *stimulated by Rota's methods*. A crude way to measure this is a citation search,

(48) Dhombres, 2003, 157.
(49) Dhombres, 2003, 157.
(50) See Rota (1969, 1972, 1995, 1998). Cartier (1972).

> but mathematical papers tend to cite direct technical links rather than indirect links of inspiration."[51]

Finally, as a side effect, Dhombres suggests that "the need to represent Reynolds operators in different concrete situations may have fostered the development of Rota's phenomenological viewpoint for mathematics. He first explained his view in 1973b in an article ["Edmund Husserl and the reform of logic", *Explorations in phenomenology*, p. 299-305. Published also in *Indiscrete Thoughts*, Rota, 1973b]. See [also Rota, 1997b]".

Contrary to what Dhombres suggests in his yet very enlightening article[52], Rota does not look to phenomenology for "simple" explanations of the possible applications in different contexts of the Reynolds operator. Rather, this minimalist interpretation explains Dhombres' severe judgment and underestimation of the importance of Rota's interests in logic[53]. (But this is an old story, as we learn from Rota's *memories* of Fine Hall, his years at Princeton). In a deep and explicit affinity with Husserl's logical investigations, Rota was looking for new resources, which could help a methodical understanding of an implicit but necessary moment in the history and practice of science (and mathematics in particular), and which could operate — as seems necessary — or induce a deeper transformation of formal logic.

We will recall the general formal framework within which Rota builds his positive and ideal historical perspectives. This framework consists of — or at least can be expressed as — a dynamic interpretation of the logical relationship between syntax and semantics. Without such a framework, Rota's specific critical

(51) Dhombres, 2003, 157 (emphasis added).

(52) "In 1973, Rota *exploited the idea of operator averaging* in the radically different context of evaluation rings of a distributive lattice in [50] [here, under (Rota, 1973a)]. This paper is fascinating because, as Rota writes, «the method of presentation is deliberately informal and discursive. Much of the reasoning is by analogy, but not in the structuralist or Bourbakian sense, and many of the *results are merely heuristic*. The *goal is grandiose, nothing less than the 'linearization of logic'*" (Dhombres, 2003, 163).

(53) "I was *too young and too shy to have a personal opinion about Church and mathematical logic. I was in love with the subject, and his course was my first graduate course*. I could feel the disapproval around me; only Roger Lyndon (the inventor of spectral sequences), who had been my freshman advisor, encouraged me. Soon after, he himself was encouraged to move to Michigan. Fortunately, I had met one of Church's most flamboyant former students, John Kemeny, who, having just finished his tenure as a professor of mathematics, was being introduced — by Lefschetz's gentle hand — to the philosophy department." (Rota, 1997b, p. 7).

stance and his statements on the past and present history of science would be difficult to understand. Following Rota, let us call it *ideal history* (*history as it should or could have been if...*). This counterfactual history is combined with a vision of history *as it really happened*, to obtain a sharper vision of the latter as well as a richer vision of what should and could happen in the future. Ideal history is the exact opposite of ideological history. From this point of view, Rota's repeated, sharp and witty criticisms of the prejudices of different kinds of academics (mathematicians, philosophers, logicians, A.I. researchers, etc.) express neither an indictment of intellectual sportsmanship nor a form of socio-psychological complex. If one had to attribute to it any psychological disposition, one would have to label it with its beautiful Platonic name: the concern for the *communication of forms*.

From this point of view, Rota's generalization of Reynold's identity is a perfect example of his strategy of radical, systematic and rigorous epistemological *decompartmentalization* (between probability theory, set theory, first order logic, etc. and philosophy and mathematics).

2. *Construction of the norm*

In a collaboration with David Sharp and Robert Sokolowski, *Syntax, Semantics, and the Problem of the Identity of Mathematical Objects* (Rota, 1988), adopting a framework similar to that of Husserl's eidetic, Rota extends the scope of cryptomorphisms and creates a tension between the two formal categories: that of semantics and syntax.

Rota's article that we will follow here traces the various stages of the algebraization of logic. Starting from Boolean algebra (the algebraic version of propositional calculus), it reviews the progress of the algebraization of logical semantics: from the algebra of sets, which constitutes the "paradigmatic example" for mathematical logic, to Birkhoff's lattice theory (1967), which represents, at least for a time, the model (*ideal example*) of a "*successful formalization of a mathematical theory*" as well as of a successful mathematical theory. Indeed, let us repeat with Rota the list of criteria that are satisfied: (1) "the *syntax of it is specified by intuitive rules of inference*"; (2) "it further includes a *complete description of all the models of such a syntax*, in this case sets", (3) which is more "*a table of semantic concepts and syntactic concepts*", and "a clear *algorithm of translation between the two*".

After the success of the series of attempts to algebraize logic, what was only a non-formalized program with a hegemonic aim has imposed itself as the norm of all semantics and all syntax. This normative claim presupposes, as the previous characterization suggests, (1) a criterion *of syntactic equivalence* of different axiomatics of the same theory *T,* which Rota calls "cryptomorphism"; (2) a *classification of the models* of this theory *T* independent of the choice of specific axiomatics; (3) a *completeness theorem* establishing a relation between "truth in all models" (semantic truth) and "truth as a demonstration from the axioms" ("syntactic truth"). If this program had succeeded, we could have immediately concluded that it is impossible to pose a mathematical object in excess of its axiomatizations and, consequently, that the project of a reform of logic[(54)] has failed, is useless or impossible.... The principle (3), called "completeness", is not what gives the familiarity with a model either, but rather it presupposes it. *Cryptomorphisms* are most often discovered after the fact, i.e., once *the mathematician has acquired a sufficient familiarity of models and a full understanding of the "eidetic identity" of models*. We come here to Rota's illuminating conclusion already quoted above (Rota, 1988, p. 383).

By "identity" or "object", Rota does not mean a particular equation or structure, but rather a *pole-idea*. Taking a stance which is opposed (and complementary) to the current understanding of model theory, as well as category theory, Rota asserts that a *single* mathematical "identity" ("object")[(55)] is itself the "*identity*" of an infinite number of objectivations (and equations), bearing various syntactic *and* semantic presentations. From a historical point of view, this pre-seizure is exposed to various series of modalizations (doubts, partial seizures, incomplete or wrong seizures, etc.). In this article, Rota focuses on the pole idea of mathematical constructs, which are neither *the*, nor *a domain of* reality, according the "mathematical realism" i.e., to the so-called "mathematical Platonism". Nevertheless, it is an essential moment (component) of reality and of every object. Avoiding crystallizing reality by stuffing it with mathematical constructs ("substructuring" it in Husserl's terms), the eidetic intentional structure of scientific experience, described by Husserl, is adapted by Rota to the sphere of

(54) For a more extensive comment on this aspect of Rota's thought, see. C. Lobo, 2017.

(55) With good reason, following in the footsteps of Husserl's critique of objectivism, Rota would later refer to them as «items» instead of "objects" or "identities", claiming the "end of objectivity", i.e. "the end of the objectivist conception of experience" (Rota, 1974-1991, 138).

mathematical experience, and he sets out an *a priori* framework for the history of mathematics, which does not exclude its openness or annihilate the freedom of mathematical thought. But in this particular case, as in the general case, to put it in Husserlian terms: However great this freedom of union and categorical formation, it always has its limits governed by law.(56)

This scope is fully and explicitly assumed, since the starting point of the article is to recognize "that this duality of presentation, syntactic and semantic, is shared by all mathematical theories and is not only found in mathematical logic."(57) From the way in which the interaction between identity, syntax and semantics is regulated, we can deduce that there is neither a single syntax, nor an ultimate syntax, and that the dual relationship explored in symbolic logic, through its transfer to mathematics, is transformed into a ternary structure.

> "The analysis of the similarity of mathematical objects, which we'll call mathematical constructs, immediately points to a closely related problem, which was first recognized in the philosophy of symbolic logic, but which has a much wider scope in the philosophy of mathematics, as we explain below. A presentation of a mathematical system leading to the definition of an object is necessarily syntactic, i.e., it is given by axioms and rules of inference. Nevertheless, axioms and rules of inference are intended to characterize a class of mathematical objects made up of sets endowed with an additional structure (such as groups, manifolds, etc.). Any set endowed with such additional structure that satisfies the axioms is considered a *model* for the axioms, and the description of all such models is usually regarded as the semantic interpretation of the theory." (*Ibid*.)

The problem is thus rephrased: "how can disparate syntaxes nevertheless have the same semantics, i.e., the same model"?

The third term, "identity", is merely an idea *at a distance.* For this reason, it is precisely the one that is most immediately given, albeit confusedly. The introduction of this third term implies a softening and opening of the whole logical frame and forces the "fundamental conclusion": "that a complete description of the logical role of

(56) Sixth Logical Investigation, § 62. Husserl, 1984, p. 717; see as well: Husserl, 1970, p. 309.

(57) Rota, 1988, p. 377.

the mathematical object is beyond the reach of the axiomatic method *as it is understood today*."[(58)] Among the examples of "identities" or "similarities": the *real line* is probably the most paradigmatic of a mathematical "similarity" supporting an infinite variety of formal syntaxes and semantics, which cannot be anticipated, but must each time be constructed. For this reason, and for the sake of clarity, before turning to the case of the probability measure, let's move on to the presentation of the eidetic and phenomenological structure of the mathematical process of its determination, as sketched at the end of the article. The words "object" and "mathematical object" are to be understood here as synonyms for "similarity", "sameness", "identity" or "eidos".

> "If one accepts our discussion leading to the conclusion that two (or more) axiom systems for the real line (say) can be recognized to present the *same* real line, then one is forced to draw the following consequences:
>
> **(1)** The real line, or any other mathematical object, is not entirely given by a specific axiom system, or by any specification of a finite set of axiom systems.
>
> **(2)** The totality of possible axiom systems for the real line cannot be exhaustively explained or predicted. Every mathematical object allows for an open sequence of presentations by ever-new axiom systems, i.e., the successive development of ever-new axiom systems for what is perceived as one and the same object. Each of these systems is supposed to reveal new characteristics of the mathematical object.
>
> **(3)** Studying the real line is not a game to be played with axioms and develop the ability to draw consequences from them. On the contrary, the very choice of which properties to deduce and how the theory is to be organized is guided from the outset by a *pre-axiomatic* understanding of the real line. Without a guiding intuition, even if unverbalized, such an axiomatic theory cannot make *sense*.
>
> **(4)** Even if the concept can initially be learned by working assiduously on the axiomatic approach, this axiomatic approach will be abandoned once familiarity with it is achieved. Thus, by learning through a particular axiomatic system, one discovers a concept whose full understanding lies literally *beyond the* scope of that system." (Rota, 1988, 384-385)

[(58)] Rota, 1988, p. 384. Emphasis added.

The highlights of the structure are as follows:

1) *Identity* (*similarity*) pre-inscribed in a variety of syntaxes, although prescribed at a distance, this identity is neither a trivial identity nor an equation.

2) An infinite plurality of syntactic presentations, uncountable in advance, each new syntactic presentation revealing, in a very slow and progressive process, new formal characteristics of *the same* object, which is not a fixed and empty closed form, but rather an inexhaustible potential for ever-new forms.

3) These new forms, these new determinations motivate and lead to new constructions, and are partially conceived, i.e., partially, informally, intuitively, tacitly etc. anticipated; this is where the "tracks" come in, giving solidity and meaning to mathematical activity.

4) Although understanding the identical object goes beyond mastering the axiomatic game, a syntactic presentation provides an unavoidable framework for anyone wishing to learn something *concrete about the* mathematical object; in other words, axiomatics is a necessary but not sufficient condition.

The question arises: *what are the operations that make it possible to realize such a structure and to place axiomatics in the phenomenological role of* (*formal*) *presentations* (Darstellungen) *of an idea-at-a-distance, of an idea as a pole* in infinitum (for instance that of the "real line")? By way of comparison, let us recall what Husserl says about the objects of geometry: they are at the same time *material eidê* and *ideas in the Kantian sense,* that pole-ideas. From the outset, the real line is neither the Euclidean line (although historically presented by it), nor, of course, the line perceived by the senses (which is precisely "idealized" by the former). Both are, in various respects, merely what Husserl calls *Anhiebe-Ideen*, that is: "initial ideas" or better, *eidetic sketches* of the pole-idea. The central eidetic operation performed is the construction of a particular type of analogy, through morphic (or "anamorphic") projections in both directions, without giving preference to one form over another. These structural analogies are precisely what is named by Rota "cryptomorphism" and, in my opinion, should be compared to Husserl's use of structural analogies (among others, in his research on affective and volitional intentionality, and correlatively, on axiology and ethics in the broad sense of the term). The promotion and cultivation of this mode of circulation are underpinned by a set of presuppositions, or rather, they involve a (historically and rationally motivated) relaxation in relation to a set of presuppositions concerning the development of rationality.

Among the decisions underlying this mode of circulation are the followings.

1) The integration and correction of model theory: "Every field of mathematics has its zenith and its nadir. The zenith of logic is model theory (we dare not say what we think its nadir is). The sure sign that it is a zenith is that when we ignorant and stupid non-logicians try to read the material, we get the feeling that it should be rewritten for the benefit of a general audience.(59)" Consequently, the severe reform of the Tarskian notion of truth (motivated by Paul Cohen's forcing): "A new paradise was opened when Paul Cohen invented forcing, soon followed by the reform of the Tarskian notion of truth, which is the idea of Boolean-valued models. On certain topics, like this one, one senses that an unfathomable depth of applications is at hand, which will lead to a recasting of mathematics."(60)

2) The admission of category theory to its rightful place: "We were turned away from category theory by the excesses of the sixties when a noisy crowd claimed to be rewriting mathematics in the language of categories. Their pretensions were toned down, and *category theory* took its modest place alongside *network theory*, more pretentious than the latter, but enjoying solid support from both Western and Eastern masters. — One wonders why category theory has aroused such fanatical opposition. One reason may be that understanding *category theory requires an awareness of the analogies between disparate mathematical disciplines*, and mathematicians are not interested in stepping outside their narrow territory"(61).

3) The rehabilitation of the updated projects of algebraic logic and universal algebra: "Ever since theoretical computer scientists began to supplant traditional logicians; we've witnessed the resurgence of non-standard logics. These new logics return problems to universal algebra, with salutary effects. Anyone who believes that the theory of commutative rings is the central chapter of algebra will have to change their tune. The combination of logic and universal algebra will take over." (*Ibid.*)

4) The reasonable and fair evaluation of lattice theory despite the bad press of quantum logic with the hope of drying up the source of a perfect example of *hopeless philosophy*: the philosophy of quantum mechanics. Rota is rather severe on that point:

(59) Rota, 1997b, p. 218.
(60) Rota, 1997b, p. 218.
(61) *Ibid*. (Emphasis mine).

> "It has always been difficult to take quantum logic seriously. One malicious algebraist scornfully dubbed it 'the poor man's von Neumann algebras'. The background of lattice theory made people suspicious, given the bad press lattice theory has always had. [...] It's hard to conceive of a more accomplished example of the *philosophy of despair* than the philosophy of quantum mechanics. It was born of a marriage of misunderstandings: the myth that logic is linked to Boolean algebra and the claim that a generalization of Boolean algebra is the notion of modular lattice. Thousands of articles confirmed mathematicians' worst suspicions of philosophers. This philosophy came to an end when someone proved conclusively that these observables, which are the analogues of random variables in quantum mechanics, cannot be described by the structure of lattice theory alone, unlike random variables. — This debacle had the salutary effect of opening the field to an honest philosophy of quantum mechanics, at the same level of honesty as the philosophy of statistics (of which we'd like to see more) or the philosophy of relativity (of which we'd like to see less)."(62)

In order to demonstrate the fruitfulness of this softened syntax/semantics relationship, Rota proceeds with a kind of eidetic variation: he lists examples of various syntactic presentations of the same semantics; and vice versa; but also, cases of syntactically accomplished mathematical theories but without semantics and conversely examples of semantics devoid of syntax. Interestingly, as a paradigmatic example of a fully realized formalization of a mathematical theory, with "*an array of semantic concepts and syntactic concepts, as well as a translation algorithm between the two*", Rota once again cites the distributive lattice theory of Birkhoff (1967) and Stone (1937). He observes that examples of semantics still lacking a clear syntax abound in mathematics, because setting up a syntax requires reflection and because, unlike logicians, mathematicians are somewhat wary of logical reflection. What's more, one need to familiarize oneself with the mathematical structures involved, through exercises, to obtain a basis for such reflection, and an initial axiomatization(63).

(62) Rota, 1997b, p. 218-219.

(63) Rota/Baclawski give the following advice to their students: "The aim of this course is to learn to think probabilistically. Unfortunately, the *only way to learn to*

Unsurprisingly, Rota gives three examples of mathematical theories that, directly or indirectly, refer to probability theory and show the need to reform its current axiomatization. Let us review them briefly.

3. *The testing of the norm on the ground of some mathematical examples.*

To test this ternary relationship between *syntax, semantics* and *eidetic identity*, Rota proposes two variations: **A)** examples of mathematical objects with syntax but without a clear semantic structure; **B)** examples of mathematical objects with semantics, but whose syntax remains obscure. Let us recall the major lines of these analyses that we have already developed elsewhere.

A) Among the examples of mathematics with syntax but without clear semantic structure. Rota mentions as a first example: The Hilbert operator ε. This is undoubtedly a *semantic operator*, but one that cannot be formalized. This notion is that of ε: the individual variable x such that $F(x)$ is true, if it exists", where $F(\varepsilon(F(x)))$ replaces the usual existential proposition $(\exists x)F(x)$[64]. On this point, there were attempts shortly afterwards, in 1993 and following years. The second example is that of the semantics of intuitionistic logic and modal logic, which until the work of Kripke were not understood. The discovery of their semantics led to a breakthrough; it was realized that intuitionistic logic was a *syntactic presentation* of the [semantic] idea of *forcing* due to Paul Cohen.[65]

B) Examples of mathematics with semantics, but whose syntax remains obscure. In general, this is the most common case in mathematics. Because syntax implies a logical reflection, and this reflection the mathematician does not like to do. Not to mention that one must first become familiar with the structure for such reflection to take place and have a chance of succeeding. The existence of a mathematical logic which develops a syntax before its semantics does not contradict this order, because the logic has been based on the familiar experience of the language, and the repeated reflections to grasp its syntactic structure.

Example 1 (transitional, as this is already a case of semantics without syntax): Closed subspaces of a Hilbert space that are the *analoga* of events in probability theory, or propositions in the classical

think probabilistically is to learn probability theorems. It is only later, after the theorems have been mastered, that the probabilistic point of view begins to emerge as the specific theorems fade from memory: rather like the Cheshire Cat's smile." (Rota/Baclawski, 1979: vii).

[64] Rota's notation instead of: $\mathrm{F}(\varepsilon_x \mathrm{F}x)$.

[65] cf. Fitting, M. C. 1969, *Intuitionistic Logic. Model Theory and Forcing*, North-Holland, Amsterdam.

sense. These Hilbert spaces are, in standard approaches to quantum physics, like the *elementary semantics* of quantum mechanics. But it is a semantics without syntax. Rota recalls here the failures of "quantum logic": "*the numerous attempts to develop a 'logic of quantum mechanics' have failed because nobody has been able so far to develop a syntactically efficient presentation of the mathematical structure consisting of the events of quantum mechanics*"[66] — the attempts of Birkhoff and Von Neumann in 1936 (modular lattices) or that of Loomis (in 1955). Ortho-modular lattices have not been admitted because of the discrepancy between their semantics and the one suggested by the current practice of quantum mechanics.

Example 2: The theory of multisets, to which Rota will have recourse in his introductory lessons on probability in 1976.[67] A multiset of a set is a generalization of a subset of S, whose elements can occur several times (thus greater than 1). Hence the definition: *a multiset is a function* (*called "multiplicity"*) *of the set S in the non-negative integers.* We can add and multiply multisets. We have an analog of a Boolean algebra for sets. But the duality between the algebra of sets and the algebra of multisets remains to be elucidated syntactically.

Example 3: The theory of probability whose presentations are syntactically informal. For example, when a probability theorist or statistician speaks of a *sequence of random variables forming a Markov chain*, he reasons directly about them without worrying about the complex construction of a phase space or state space.

> "The statistician who computes with confidence intervals and significance levels seldom appeals to the measure-theoretic justification of his reasoning. It can be surmised that a syntactical presentation of probability will view joint probability distributions as playing a role similar to truth-values in predicate calculus. Kolmogorov's consistency theorem shows how to construct actual random variables in a sample space whose joint distributions are formalized to be a given family of consistent distributions. Thus, from this point of view, Kolmogorov's theorem *should turn* out to be the completeness theorem relating the syntactic and semantic presentations of probability theory".[68]

[66] *Op. cit.* p. 380.

[67] Rota (Gian-Carlo), Baclawski (Kenneth), 1979, An introduction to probability and Random Processes.

[68] Art. cit. p. 381. This example has been presented more thoroughly in Rota, 2001. See my comments in Lobo, Some Reasons to reopen the Question of

§ — Conclusion.

Rota's practice of a history of mathematics, sensitive to the eidetic structure of mathematical experience has manifest and declared affinities with Husserl's historical considerations developed in the *Krisis*. Both are opposed to ideological exploitations from artificially reduced or blind perspectives, whose various individual and sociological symptoms are for Rota as for Husserl: *reductionism* and *objectivism*.

Unless we consider the ternary relation between syntax, semantics and identity, and identity as a moment of an eidetic structure, with its opened horizon structure, mathematical experience and historicity remain unintelligible. But taking them into account requires adequate concepts and analytical resources. In particular, the codetermining modes belonging to the domain of modalizations are essential.

Of course, Rota himself insists on the "empiricism" of such an attitude and refers it back to Lakatos. But this "empiricism" in brackets is, like the epistemic "relativistic" sensibility, the expression of an epistemic sense of *contextual dependence* of all identity, of all *idealities*. Far from falling into some form of relativism, historicism, sociologism, this sensitivity of knowledge to context is a new and deeper vision of the *eidos*, and *eidos* that is at the heart of "the inner historicity of mathematics as a science."(69)

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(69) Art. cit. p. 384.

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