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Mathematics
an Imagined Tool for Rational Cognition
Part I

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"...numbers are free creations of the human mind;
they serve as a means of apprehending more easily and more sharply the difference of things."

Richard Dedekind [Dedekind, 1888, p. vii]

Abstract. By analysing several characteristic mathematical models: natural and real numbers, Euclidean geometry, group theory, and set theory, I argue that a mathematical model in its final form is a junction of a set of axioms and an internal partial interpretation of the corresponding language. It follows from the analysis that (i) mathematical objects do not exist in the external world: they are imagined objects, some of which, at least approximately, exist in our internal world of activities or we can realize or represent them there; (ii) mathematical truths are not truths about the external world but specifications (formulations) of mathematical conceptions; (iii) mathematics is first and foremost our imagined tool by which, with certain assumptions about its applicability, we explore nature and synthesize our rational cognition of it.

Keywords. Mathematical intuition; mathematical models; mathematical objects; mathematical truths; applicability of mathematics

The basic problem of the philosophy of mathematics (not mathematics itself) is to answer the following intertwined questions:

- Whether mathematical objects exist, and if so, in what way do they exist?
- What is the mathematical truth and how do we establish it?
- How is mathematics applied?

This article and its sequel present a solution that can be considered an elaboration of Dedekind's quotation cited at the beginning of the article. The basic theses that I intend to argue are the following:

- Mathematical objects do not exist in the external world. They are imagined objects, some of which, at least approximately, exist in our internal world of activities or we can realize or represent them there.
- Mathematical truths are not truths about the external world but specifications (formulations) of mathematical conceptions.
- Mathematics is first and foremost our imagined tool by which, with certain assumptions about its applicability, we explore nature and synthesize our rational cognition of it.

I will try to make clear what is absolutely clear to the famous physicist Percy W. Bridgman: "It is the merest truism, evident at once to unsophisticated observation, that mathematics is a human invention." [Bridgman, 1927, p. 60]. Having practised mathematics all my life, by vocation and by profession, just as breathing was natural to me, so it was natural for me to consider mathematics as a human invention and a free creation of the human mind whose purpose is to be a tool for our rational cognition and rational activities in general.(1) When I decided to clarify to myself what human invention mathematics was, it proved to me, I believe, a far more difficult task than explaining what breathing is. In this article and its sequel, I have outlined what I came up with along the way.

In this article I will consider as illustrative examples the classical mathematical models which are still the most important ones: the natural number system, the real number system, and Euclidean

(1)Of course, many mathematicians do not share my opinion that mathematics is a free creation of the human mind.

geometry; as well as contemporary standard mathematical models: group theory, and set theory (sections 1 to 6). To quote the famous mathematician Saunders Mac Lane: "...a philosophy of Mathematics is not convincing unless it is founded on an examination of Mathematics itself." [Mac Lane, 1986, p. 60]. In the last section (Section 7) I will set out the basic characteristics of the view of mathematics as an imagined tool for rational cognition. In the next article, I will elaborate in more detail how mathematical objects possibly exist and how mathematics is applied. I will also describe how the whole structure of mathematics can be understood as an imagined tool for rational cognition. Also, there I will expose this view of mathematics to various tests as well as determine its place on the map of different views of mathematics. In [Čulina, 2022a] there is a complete exposition in the form of a preprint.

The terms "intuition", "idea", "conception", "model" and "theory" will denote more and more precise stages in modelling certain thoughts. The terms "model" and "theory" will eventually acquire a precise meaning that deviates from the standard meaning in mathematical logic. They will be synonymous here and will denote a set of axioms in a language, together with a partial interpretation of the language. In doing so, I will give preference to the term "model" when the emphasis is on the interpretation and to the term "theory" when the emphasis is on axioms. Thereby, the term "interpretation" generally refers to the meanings of linguistic forms, and more specifically to the interpretations of first-order languages. The term "specification" generally refers to the refinement of ideas, and more specifically to the description using a set of sentences (axioms) of a first-order language that an interpretation of the language must satisfy. By "internal world of activities" I mean the world that consists of activities over which we have a strong control, and which we organize and design by our human measure. For example, these include movements in a safe space, grouping, arranging and connecting small objects, spatial constructions and deconstructions with small objects, talking, writing and drawing on paper, shaping and transforming manipulative material, combining and repeating actions, making choices, dynamics of actions and changes in the environment subordinated to us, painting, singing, etc. This does not include activities with objects over which we have no strong control, or our activities are significantly limited by the environment, such as e.g. climbing steep rocks, building a house, etc. Although each person has her own specific way of organizing and designing internal activities, which depends on her as well as on the environment to which she belongs, due to the generality of further considerations, it does not matter whether we mean the internal world of activities of an individual person or the internal world of activities of the entire human species. The internal world of activities will be discussed in more detail in Section 7. I would also note here that all imagined objects are considered as concrete and not abstract objects. In Section 7 it will be argued that language is the bearer of abstraction and not objects.

§ 1. — Natural numbers.

Natural numbers are the result of modelling our intuition about the size of a collection of objects. We measure the collection through the process of counting, and natural numbers are objects for counting. To start counting we must have the first number, to associate it with the first chosen object in the collection. To continue counting, after each number we must have the next new number in order to associate it with the next chosen object in the collection. There is no special reason to sort out certain particular objects as natural numbers. Merely for the needs of performing a calculation we sort out a particular realization. In the past those were collections of marbles on an abacus, and today we use sequences of decimal digits on paper and of bits in a computer. It means that for counting it is not important how numbers are realized, but only the structure of the set of natural numbers which enables us to count is important: that there is a first number and that each number has its successor. It seems that natural numbers exist in the same way as chess figures, in the sense that we can always realize them in some way. However, the structure of natural numbers, as opposed to the structure of chess, brings in itself an idealization. So that the counting could always continue, each natural number must be followed by the next natural number. Therefore, there are infinitely many natural numbers. Thus, although we can say for small natural numbers that they exist in the standard sense of that word, the existence of big natural numbers is at best a kind of idealized potential existence.

In order to precisely formulate the conception of natural numbers, we need a corresponding sufficiently precise language, a mathematical language, or even better, a first-order language. Among other symbols, the language should contain the symbol for the first number, the standard is "1", and the function symbol, for example an "S", for the immediate successor operation ($n \mapsto \mathrm{S}(n)$). This allows us to name each number. What the names denote is not so important. It is only essential that the named objects have the structural role of natural numbers (the names can even denote themselves). We specify the conception by declaring certain sentences of the language to be true about natural numbers. It is not necessary to precisely specify these claims here, nor the language in which they were made.(2) In that language, the claims must express, inter alia, (i) that 1 is the first number, (ii) that each number n has its successor $\mathrm{S}(n)$ which is a new natural number in relation to all previous natural numbers, and (iii) that any natural number can thus be obtained. I will hereinafter call these claims the axioms of natural numbers. In my view, the axioms of natural numbers are neither true nor false, just as the axioms that would describe the game of chess would be neither true nor false. They are simply a means of specifying our ideas about the objects we use for counting.(3) However, Gödel's incompleteness theorems [Gödel, 1931] and Lowenheim-Skolem theorems [Löwenheim, 1915; Skolem, 1920] tell us that we cannot have a complete (at least up to isomorphism) specification in a first-order language, even if we include sets in the specification.(4) Therefore, in addition to the axiomatic specification, it is necessary to have an interpretation of the language, as well. In the case of natural numbers, it is a partial internal interpretation — a partial interpretation in our internal world of activities. The interpretation is partial due to the idealization of the existence of extremely big numbers. The interpretation belongs to our internal world of activities

(2) The standard formulation consists of Peano's axioms and recursive conditions for addition and multiplication in the first-order language of arithmetic [van Dalen, 2013, p. 82].

(3) In [Ferreirós, 2016, Chapter 7] Ferreirós argues that Peano's axioms are self-evident truths about our practice of counting. Our practice of counting is not a natural process but a voluntary activity. That is why, in my opinion, the truth of Peano's axioms does not derive from that practice, but from the fact that they norm that practice, especially since they introduce an element of idealization into the practice.

(4) Although in the classical set theory all structures of natural numbers are mutually isomorphic, according to the Lowenheim-Skolem theorems, the classical set theory itself has non-isomorphic interpretations [van Dalen, 2013, p. 105-106].

because numbers are our imagined constructions that we can partially, to the point of isomorphism, realize in the world available to us: using marbles on an abacus, tallies on paper, etc. On the other hand, even in the idealization of the total interpretation of the language, it is necessary to additionally have an axiomatic specification, because the recursively defined truth value of sentences in the interpretation is not a computable function [Gödel, 1931; Tarski, 1933]. Since we can never construct all numbers, the overall structure of natural numbers does not exist in the literal sense of the word. There is only the conception of natural numbers, specified by axioms, and partially realized in our internal world of activities. This is the final result — mathematical model — of modelling our intuition about natural numbers as objects for counting. It carries with it incompleteness and the ever-present tension in mathematics between basic intuition and the constructed model, as well as between the axiomatic specification and the constructive content of mathematical concepts. This tension is a positive source of accepting new axioms as well as improving intuition.(5)

§ 2. — Real numbers.

Through real numbers we organize and make precise our intuition about the process of measuring.(6) While natural numbers are imagined as objects for counting, real numbers are imagined as the results of the process of measuring. However, we can imagine (idealized) situations in which the process of measuring never stops — we generate a potentially infinite list of digits, with no consecutive repetition of the same group of digits after an individual step. If we want to have the results of such processes of measuring, we must introduce, in addition to rational numbers, new results of measuring — irrational numbers. As opposed to natural numbers whose existence

(5) This tension took a more dramatic form in the historical conflict between modern and classical mathematics, primarily in the confrontation between Dedekind's modern conceptualist approach and Kronecker's classical constructivist approach, as well as in the crisis in the foundations of mathematics — in the conflict between Hilbert's program in the foundations of mathematics and Brouwer's intuitionism (see, for example, [Ferreirós, 2008]). How much of this tension is present in the Hilbert's program itself can be read in [Sieg, 1999].

(6) In my opinion, this is the main role of real numbers, as a tool for rational cognition. Various practices with real numbers and their central role in mathematics are nicely described in [Ferreirós, 2016, Chapter 8].

we can understand at least as some kind of an idealized potential existence, we cannot explain the existence of irrational numbers in this way. Although we can approximate irrational numbers by rational numbers with arbitrary precision, their existence is outside our means of construction — we have just imagined irrational numbers.

As is the case with natural numbers, the final mathematical model of real numbers is a junction of axiomatic specification(7) and partial internal interpretation of the corresponding language. For example, in the mathematical model we can identify Euler number e, the irrational number to whom the sequence $\left(1+\frac{1}{n}\right)^n$ is closer, as we increase the natural number n. Although we can approximate the number e with arbitrary precision by constructions in our internal world of activities, it certainly does not exist in the same way as my dog exists. It exists in the same way as an idealized material point in classical mechanics, as a non-existing phlogiston in a wrong theory about chemical reactions, and as Snow White in the classical fairy tale *Snow White and the Seven Dwarfs*. However, although our language usually has only a partial interpretation, the classical logic of using the language assumes that it is a semantically complete language — that it has a complete interpretation: each name names an object, each predicate symbol refers to a binary predicate, and each function symbol refers to a function.(8) Because of this assumption, in thinking itself there is no difference whether we think of objects that really exist or we think of objects that do not really exist. That difference can be registered only in a "meeting" with reality. And for mathematics there is no such meeting: a mathematical model creates its own reality in our internal world of imagination. However, unlike erroneous physical models, the mathematical model of real numbers can be realized approximately in our internal world of activities in the same way that correct physical models are realized approximately in the external world or children's fairy tales in real theatrical performances.

(7) The axiom of completeness ensures that any idealized measurement process has a result. A variant of the axiom corresponding to this approach postulates that every decimal expansion $a_0, a_0.a_1, a_0.a_1a_2, \ldots$ (where $a_1, a_2, \ldots$ are decimal digits, whereas a_0 is an integer) has a limit and that every real number is a limit of such an expansion. All other variants of the completeness axiom postulate in other ways these imagined objects. See, for example, [Deveau and Teismann, 2014] — the variant with decimal expansions has the label CA18.

(8) For complete interpretations of the first-order languages see, for example, [van Dalen, 2013, Section 3.4]

§ 3. — Euclidean geometry.

With the appearance of non-Euclidean geometries in the 19th century, Euclidean geometry lost the status of an a priori mathematical theory. It became only one of the possible models for physical space, distinguished only by the fact that it is a good approximation of the space in which practical science takes place.(9) Contrary to such a view, according to which Euclidean geometry is a part of physics, I will argue here that just as number systems are idealized conceptions derived from intuition about our internal activities of counting and measuring, so too is Euclidean geometry an idealized conception derived from intuition about our internal spatial activities. Since the view of Euclidean geometry presented here is not standard, the argumentation will be more detailed than in the sections above.

Internal spatial activities should be distinguished from external spatial activities. The former are conditioned by our human nature, the latter additionally by the world around us. At first glance, it seems difficult, almost impossible, to draw a clear line between these two types of activities. However, for some activities we can clearly determine that they belong to external spatial activities. For example, mountain climbing is an external spatial activity because it involves orientation in a given landscape and taking care of the configuration of the terrain over which we move. When we are on the different parts of a mountain road, different spatial situations will require different responses. However, as far as our ability to react is concerned, it is the same in all places and in all directions. Of course, there is also the ubiquitous gravitational force that makes one direction in space prominent, which we especially have to take care of. Likewise, if there is a strong wind blowing from a certain direction, that direction also becomes prominent to us. However, we always attribute the appearance of differences in a certain direction to the influence of an external factor, which shows that a priori all directions are the same to us. If we have to light a fire by placing twigs so that they form a cone, our approach to geometric construction will be the same, whether we are making a small or a large cone. This shows that our ideas of spatial constructions are independent of the units we use in their construction. If for a certain selection of units a change in construction occurs, we ascribe

(9) In [Torretti, 1978], the impact of the appearance of non-Euclidean geometries on the philosophy of geometry is analyzed in detail.

it to an external factor. If in this way we try to identify the nature of our internal spatial activities, as invariants to the different spatial situations in which we find ourselves, then we are very close to Delboeuf's analysis [Delboeuf, 1860]. He considers what remains when we ignore all differences of things caused by their movements and mutual interactions. According to Delboeuf, in the ultimate abstraction from all diversities of real things we gain the homogeneous (all places are the same), isotropic (all directions are the same), and scale invariant (geometric constructions are independent of size) space — the true geometric space which is different from the real space. However, for Delboeuf this geometry is the background geometry of real space, while for me it is the geometry of our internal activities in space.

We can come to the same conclusion if instead of an external argument, seeking common ground in all our external spatial activities, we use an internal argument, analysing our internal spatial activities directly and independently of the external world. A simple introspection shows that we do not distinguish different places, different directions and different units for spatial constructions until the outside world forces us to distinguish them. To eliminate the presence of gravity on the Earth's surface we must look for examples where it is negligible. In addition to the extravagant situation of a free fall, these can be examples of activities that take place approximately in the horizontal plane or three-dimensional examples in which gravity is not important. For example, a child will build his imaginary monster using Lego bricks regardless of the location where he built it, the way he oriented it in space and the dimensions of basic Lego bricks used in its construction. The same indifference to location, direction, and size is present when we rearrange the Rubik's cube.

I believe that our most basic approach to space, the approach inherent to us, is an *a priori ignorant approach to space*: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and all units of length we use for constructions in space are the same to us (the scale invariance of space). These three principles of symmetry express our basic intuition about our internal spatial activities. Any deviation from these symmetries we attribute to the external world. Thus, it is precisely these principles of symmetry that determine a clear boundary between our internal and external spatial activities.(10)

Just as internal and external spatial activities can be distinguished, so can the geometry resulting from these activities. In my view of mathematics as our, in a certain sense, a priori tool of rational cognition,(11) geometry arising from internal activities is a part of mathematics, while geometry arising from external spatial activities is a part of physics. For example, caring about the direction of gravity belongs to external spatial activities, so pointing out that direction belongs to physical geometry and not to geometry arising from our internal spatial activities. Of course, in this physical geometry, as in other physical theories, appropriate mathematics is incorporated, but it is a different tool of rational cognition (see Section 5) than the geometry that arises from our internal spatial activities. In the book [Schemmel, 2016], it is nicely described how appropriate physical geometries emerge from our external spatial activities, which are very important for our survival.

In [Čulina, 2018], an elementary system of the axioms of Euclidean geometry is developed. On the one hand, the system is directly founded on the three principles of symmetry described above, while on the other hand, through the process of algebraic simplification, it gives an equivalent Weyl's system of axioms of Euclidean geometry (the axioms of Euclidean affine space) [Weyl, 1918, Chapter 1]. In this way, Euclidean geometry is characterized by these three principles of symmetry without any additional assumptions (except the idea of continuity). Thus, I gave the argument that Euclidean geometry is an idealized mathematical model derived from intuition about our internal spatial activities.

I consider this interpretation in space of our human internal activities the primary interpretation of Euclidean geometry. However, we can preserve the sentence part of the theory but

(10)The importance of these principles has been recognized a long time ago in the works of Delboeuf [Delboeuf, 1860], Helmholtz [Helmholtz, 1868], Clifford [Clifford, 1873, 1885] and Poincaré [Poincaré, 1902], but in a different interpretation than the one described here. In the 17th century John Wallis proved, assuming other Euclid's postulates, that the scale invariance principle saying that "For every figure there exists a similar figure of arbitrary magnitude." is equivalent to the Euclid's fifth postulate [Wallis, 1695-1699]. Also, it is well known that among all Riemannian manifolds Euclidean geometry is characterized by these three symmetry principles (see, for example, [Clifford, 1873]).

(11)I will explain my views below in Section 7.

change the interpretation. Then it does not need to be a mathematical conception anymore. It depends on a new interpretation, be it an external or an internal one. If we ask ourselves whether the physical space obeys the axioms of Euclidean geometry, we must extract from space what we consider as points (maybe enough localized parts of space), as directions (maybe directions of light rays), and the distance between two points (maybe the time needed for light to pass from one point to another). If in such an interpretation the physical space satisfies the axioms of Euclidean geometry then we have an experimentally verifiable theory. Its sentence part is the same as in our mathematical theory of the space of our human activities, so we can transfer all results to the structure of physical space. Only the interpreted part is different. It does not belong to mathematics anymore, but it is a base for an experimental verification of the theory about the external world. However, we can change an interpreted part of the originally imagined Euclidean geometry in a way that it will be still a mathematical theory. And it happens in mathematics often. Namely, when we investigate complex mathematical objects which we cannot perceive so easily, for example a set of functions of some kind, it is useful to find Euclidean structure in it. Then we can transfer our geometric intuition to that set — think of functions as points, measure how distant two functions are, etc. In that way we can visualize them and succeed in thinking about them more easily and effectively.

The example of Euclidean geometry purports that only the axiomatic part of a theory can belong to mathematics, while the interpretation does not have to. Also, mathematical interpretation does not have to be an idealized direct interpretation in our internal world of activities, but it can also be an interpretation in another mathematical model (theory).

§ 4. — **Group theory.**

Group theory, in addition to being an elementary part of more complex mathematical theories, models above all our intuition about symmetry as invariance to certain transformations. Since different situations have different symmetries, unlike previous mathematical models which have an intended interpretation, this theory does not have an intended interpretation, but has intended non isomorphic interpretations. Thus, some mathematical models are simply sets of axioms without a specific interpretation. However, if they are modelling some important inner intuition about our approach to the world, as group theory does, then they are usually very important. Probably, the most famous example is the Riemann's conception of geometry as a manifold with a metric [Riemann, 1883]. These models found their application half a century after their invention with the appearance of Einstein's general theory of relativity. Today, manifolds are an essential component of mathematics and physics. Although the application was realized so late, it had to happen, because manifolds model successfully the basic mathematical idea about the coordinatization of investigated objects, an idea that generalize such an efficient idea of measuring. Although, due to their generality, the theory of groups and the theory of Riemannian manifolds have no interpretation in our world of internal activities, they grew out of intuition about the world of internal activities, the former on the ideas of transforming objects and combining transformations, the latter on the idea of coordinatization of objects.

In the language of group theory, due to the existence of non-isomorphic interpretations, we sometimes think of a definite, and sometimes of an indefinite ("any") interpretation. However, the very use of language and its logic requires that when we think in a language, we necessarily assume that it is a semantically complete language, no matter how we imagine the interpretation. The situation is the same as when we use variables in our thinking. Whether we attach a certain value to a variable or not, in thinking within a classical mathematical language we necessarily assume that it has a certain value. However, if we think of groups in the language of set theory, then the groups themselves are the values of variables, not interpretations of the whole language as described above, and we think of them differently. The language of set theory allows us to connect and compare groups with each other, without having to know the true nature of individual groups, but possibly their isomorphic copies in the world of sets without urelements. Thus, the process of modelling the initial intuition and the way of working with the constructed models depends on the language in which we model the intuition.

§ 5. — Mathematical models from other disciplines.

The source of mathematical models does not have to be an intuition about our internal world of activities. They can also be "borrowed" from other disciplines. The nature of our thought and use of language, as well as the way in which we manage the vast complexity of the world, leads to the extraction of a certain structure from such a domain. We extract from the domain certain objects, relations and operations and we describe their properties. If we have thus obtained an important model from that domain then its sentence (axiomatic) part is a mathematical model important for examination. We can use classical mechanics to illustrate this point. Although particles, motion and forces do not belong to mathematics, mathematics can take the structural properties of phenomena (usually described as a set of sentences in an appropriate language) and formally investigate them: the consequences (for example, in the problem of three bodies), the equivalent formulations (for example, Lagrangian and Hamiltonian formulations of Newtonian mechanics emerged in this way), etc.

§ 6. — Set theory.

In the consideration of any objects, the consideration of the sets (collection) of those objects naturally occurs. In mathematics, this step has a deeper meaning. Namely, the foundational mathematical modelling must model the very intuition about mathematical and thought modelling itself. In the process of thought modelling, we extract a structure from a set of objects, that is to say, we extract some distinguished objects, and some relations and functions over the set of objects. Therefore, the subject of the foundational modelling must be the structures themselves and their parts. We can reduce the description of the structures to the description of their parts. It is the standard result of mathematics that we can describe distinguished objects by functions, functions by relations, and relations by sets. In this way we can reduce the foundational modelling to the analysis of sets. From sets we can build all structures. Also, we can compare such structures using the functions between them. The language of sets provides simple means for describing structures and constructing new structures from the old ones. In this way, sets give us an universal language for mathematics. Furthermore, sets are often necessary for specification. For example, the specification of natural numbers requires the axiom of induction, which, in its most general formulation, needs the notion of a set. Likewise, the specification of real numbers requires the axiom of continuity, and the specification of Euclidean geometry requires Hilbert's axiom of maximality, and they both need the notion of a set. However, in what way exist the set of natural numbers, the set of real numbers, and the set of space points, when they are infinite? Moreover, when we think of sets, we also consider sets of sets. If we want to have an elegant, rounded and universal set theory, infinite sets are naturally imposed on us, truly the whole infinite hierarchy of infinite sets, together with infeasible operations on them.(12) How can we understand the existence of such sets and operations? Should we reject this theory, which has proven to be very successful, because its objects can be realized only when they are finite? Hilbert, who certainly knew what good mathematics is, said the following on the Cantor's set theory of infinite sets:(13) "This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of purely rational human activity," [Hilbert, 1926, p. 167].(14)

The language of set theory presupposes an intended interpretation. In addition to the fact that we can only partially realize it, the interpretation itself is not clear to us in many ways. It is clear that the idea of a set derives from our activities of grouping, arranging and connecting objects and that set theory is an idealized mathematical model for these activities. However, in the very finite part, when we talk, for example, about the set containing three concrete objects o_1, o_2 and o_3, it is not clear what kind of object the set itself is, let us call it $s = \{o_1, o_2, o_3\}$. Formally, we can describe the situation by saying that we have added a new object to the objects, which we then call the set of these objects and which has the unique property that only the objects o_1, o_2 and o_3 and no other belong to it. Such a description would correspond to a combinatorial approach and obviously has a structuralist overtone — the very nature of sets is not important but their relationship to other objects is important. However, another description $s = \{x | x = o_1 \text{ or } x = o_2 \text{ or } x = o_3\}$ of the same set has a different connotation. Now the set is given by

(12) An elegant exposition of the standard ZFC set theory can be seen, for example, in [Enderton, 1977].

(13) Cantor's exposition of his set theory can be seen in [Cantor, 1883].

(14) The complicated and at times dramatic historical development of set theory is described in detail in [Ferreirós, 2007].

a predicate, so it is a kind of extensional abstraction of an one-place predicate. In this view of sets, as extensions of one-place predicates, they have no structural role but have their own individual nature in our world of meaningful linguistic forms, in the same way that points and directions have their own individual nature in our world of internal spatial activities. Let us note that both the structural and individual view of sets change the initial intuition about the impossibility of realizing infinite sets. In the structural view, the set of all natural numbers is simply a new object. It differs from a finite set only in that there are infinitely many objects in the membership relation with it. Natural numbers can be defined without the concept of infinity.(15) Thus, from a structural point of view, the formation of the set of all natural numbers is not burdened by the concept of infinity, although it has infinite members. In the individual view, the predicate "to be a natural number", as commented above, can be defined without the concept of infinity. Let's compare, for example, that predicate with the predicate "to be an electron". The letter predicate can be defined by a certain experimental procedure. It is well defined regardless of whether there are finitely or infinitely many electrons in the world. It is the same with the predicate "to be a natural number": it is well defined regardless of whether there are finitely or infinitely many natural numbers. In the individual point of view, that predicate, as well as any other that has the same extension, can be considered the set of natural numbers. So, we can conclude that the formation of the set of natural numbers is not burdened by the concept of infinity, although it has infinite members. Thus, both in the structural and in the individual view, the set of natural numbers is a finitely well-formed object. The idea that a set must be "made" of its elements is not present here at all, an idea according to which we can never make an infinite set. Despite all the doubts related to the notion of a set, the constructed mathematical model is very successful. Today, ZFC axioms form its axiomatic part and the model has an intended interpretation, although there are doubts as to what the interpretation is, what we can realize and how we can realize it. Here we only have a more pronounced tension between the basic intuition and the final model, which can ultimately lead to model refinement, model change, or even separation into multiple models.(16)

(15) see, for example, [van Orman Quine, 1963, p. 75]

(16) In addition to other models of set theory that complement, weaken or change the ZFC axioms (see [Fraenkel and Bar-Hillel, 1958] and, for newer alternatives,

§ 7. — What mathematics is.

The conceptions described above possess all of the essential characteristics of mathematical conceptions. First of all, mathematical conceptions have a clearly defined purpose — to be successful tools in the process of rational cognition, and in rational activities in general. This purpose significantly influences their design and determines their value. We can use mathematical conceptions directly, like the use of numbers, through an ideal model of interaction with the world. We can use them indirectly: (i) like the use of the Euclidean geometry — by changing the interpreted part of the theory into an external interpretation, (ii) like the use of the group theory — by giving just the axioms and their consequences regardless of interpretation, which could be the external one, or (iii) like the use of the set theory — by organizing effectively other mathematical tools. Also, we use mathematical conceptions indirectly, (iv) as constituent parts of more complex mathematical conceptions — as it is, for example, the case with Euclidean space as a tangent space on a Riemannian manifold, or (v) we use them indirectly in the way described above with Euclidean geometry — to interpret them in collections of complex mathematical objects for the purpose of making them more intuitive and more manageable. A multitude of specific mathematical models that are used to model specific problems should also be mentioned here. Such a model is applied directly, its purpose is concrete, and its design and evaluation largely depend on the problem it models.(17)

The conceptions described above purport that mathematics is an inner organization of rational cognition and knowledge, a thoughtful shaping of the part of the cognition that belongs to us. For example, we organize the possible results of measurement into an appropriate number system. The inner organization needs to be distinguished from (but not opposed to) the outer organization of rational cognition, a real shaping of an environment that comprises construction of a physical means for cognition (for example, an instrument for measuring temperature). Mathematics is a process

[Apostoli et al., 2009]), the multiverse view in set theory has recently been developed, according to which there is no "true" mathematical model for the concept of a set, but there are various equally acceptable models [Hamkins, 2012].

(17) In the book [Carrier and Lenhard, 2017], mathematics is analysed as a means of building specific mathematical models, while in the book [Fenstad, 2018] a general model of such use of mathematics is given.

and a result of shaping our intuitions and ideas about the reality of our internal activities, into thoughtful models which enable us to understand and better control the whole reality. For example, we shape our sense for quantity into a system of measuring quantities by numbers. Thoughtful modelling of other intuitions about our internal human world of activities, for example intuitions about symmetry, flatness, closeness, comparison, etc., leads to other mathematical models. This claim will gain its full meaning only when I explain what the terms "internal world of activities", "intuition" and "mathematical model" mean. That is the content of the rest of this section.

In the introductory part, I briefly described our internal world of activities as the world that consists of activities over which we have a strong control, and which we organize and design by our human measure. The examples I listed there are activities that we can easily recognize in free children's play.(18) These activities form the basis of the adult world of internal activities. These activities develop as we grow up, but this is primarily the development of their design and conceptualization. Their presence in early development stages indicates their biological basis.(19) They are a unique characteristic of the human species, an essential part of our evolutionarily developed abilities by means of which, unlike other species that adapt to the environment, we adapt the environment to ourselves.(20) Of course, cultural evolution and social context play a decisive role in designing and conceptualizing these activities.(21)

There are many examples of highlighting the importance of these activities. Dedekind talks about "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent

(18)In [Čulina, 2022b], it is argued that, contrary to the narrow standards of mathematics education, we best help children in their mathematical development by providing them with an environment in which they will, in free play, and with our unobtrusive help, develop their internal world of activities, design it, conceptualize it and apply it to problem-solving.

(19)The book [Lakoff and Núñez, 2000, p. 28] discusses: "ordinary cognitive mechanisms as those used for the following ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on."

(20)"Man is a singular creature. He has a set of gifts which make him unique among the animals: so that, unlike them, he is not a figure in the landscape –- he is a shaper of the landscape." [Bronowski, 1974, p. 19].

(21)See, for example, [Kitcher, 1983] for the influence of cultural evolution and [Ernest, 1997; Hersh, 1997] for the influence of human society.

a thing by a thing, an ability without which no thinking is possible" [Dedekind, 1888, p. viii]. Hilbert sees the source of his finitist mathematics in "extralogical concrete objects that are intuitively present as immediate experience prior to all thought", and these objects are "the concrete signs themselves, whose shape ...is immediately clear and recognizable" [Hilbert, 1926, p. 171]. Feferman writes that the source of mathematical conceptions "lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time" [Feferman, 2014, p. 75]. Hersh writes: "To have the idea of counting, one needs the experience of handling coins or blocks or pebbles. To have the idea of an angle, one needs the experience of drawing straight lines that cross, on paper or in a sandbox" [Hersh, 1979, p. 46]. And so on.

Internal activities are concrete activities with concrete objects: they take place in space and time, and in a given environment — on the table with a pencil and paper, in a ballroom, or on a sandy beach. They are experiential activities, but it is not an experience of the external world, but an experience of our actions in the world available to us and subordinated to us. We experiment not with the objects of the external world but with the possibilities of our actions in the world adapted to us. In this world, Piaget distinguishes two types of knowledge [Piaget, 1970, p. 16-17]. An example of the first type of knowledge is when we pick up two objects and determine that one of them is heavier. This knowledge arose from our action performed on the objects and its source is in these objects: it is knowledge about these objects — knowledge about the world. Piaget calls such knowledge "physical knowledge". But when we order objects, for example by weight, this order was created by our actions and its source is in our activities and not in the objects. The knowledge that has its source in such activities, for example that there is always the same number of objects no matter how they are ordered, is called by Piaget "logical mathematical knowledge". Mathematical knowledge has its source in the world of internal activities precisely in this way: it springs from these activities themselves and not from the objects of these activities. Feferman writes: "Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable." [Feferman, 2014, p. 75]. However, we cannot completely separate activities from the objects of the activities: generally speaking, they depend on the objects. For internal activities, it is important that this connection is weak. Our control over these activities is the dominant force, rather than the influence exerted over them by the objects.

Likewise, it is not possible to draw a clear line where the world of internal activities ends, and activities become external. An example can be made out of constructing and deconstructing objects. When a child does this with Lego blocks, it is surely an internal activity. Although the structure of Lego blocks affects the possibilities of construction, these are activities over which we have a strong control and the possibility of designing them according to our own measure. Constructing and deconstructing stone walls without mortar is certainly not an internal activity, because it requires experience working with weights, centres of gravity and forces in contact. But both activities are the source of the same mathematical idea, the idea of analysis and synthesis of what we are researching: let us examine the phenomenon by breaking it into parts, study those parts and synthesize the knowledge thus acquired into knowledge about that phenomenon. This idea is a mathematical idea because it has its source in our approach to the world, not in the world itself. Although it is present in both mentioned activities, in the internal activity it takes a clear and separate form, while in the other activity it is connected with the physical content. Because of the freedom we have in the world subordinated to us, I believe that we can always internalize external activities, represent them with internal activities in which the mathematical idea will come to its full expression.

To conclude, although it is about concrete activities in a real environment, due to the subordination of that environment to our activities and due to the strong control over these activities and the strong possibilities of their shaping, we can talk about these activities as our internal world of activities, and even about a certain kind of a priority of that world.

The concept of intuition is not an intuitive concept. In [Kitcher, 1983, p. 49] Kitcher writes: "'Intuition' is one of the most overworked terms in the philosophy of mathematics, Frege's caustic remark frequently goes unheeded: 'We are all too ready to invoke inner intuition, whenever we cannot produce any other ground of knowledge.'"(22) In [Hersh, 1997, p. 61-62], Hersh gives a whole list of different and unclear meanings of this term, but also emphasizes its central role in the philosophy of mathematics: "1. All the standard philosophical viewpoints rely on some notion of intuition., 2. None of them explain the nature of the intuition that they postulate., 3. Consideration of intuition as actually experienced leads to a notion that is difficult and complex, but not inexplicable., 4. A realistic analysis of mathematical intuition should be a central goal of the philosophy of mathematics.". However, intuition about our internal world of activity is completely clear, it is intuition in the original sense of that word — "immediate awareness" — devoid of any ambiguities and misunderstandings. In my opinion, mathematical intuition is precisely this intuition.(23)

Major mathematical models, like the models described above, arise from intuition about our internal activities and organization. It is from these concrete activities that the idea of an idealized world emerges, the world that expands and supplements the internal world of activities.(24) In his book [Mac Lane, 1986], Saunders Mac Lane describes this process on a multitude of examples. The table on page 35 of the book shows a whole list of examples of activities from which certain ideas are born, and from these ideas mathematical concepts and models arise. For example, movements contribute to the idea of change, whose formulation contributes to the concepts of rigid motion, transformation group and rate of change; estimating contributes to the ideas of approximation and closeness, and their formulation contributes to the concepts of continuity, limit and topological space. However, what is the nature of idealized mathematical models? What is the nature of the irrational numbers that Dedekind "creates" to fill the "gaps" in the linearly ordered field of rational numbers [Dedekind, 1872]? Hilbert writes

(22) For example, Brouwer in his First act of intuitionism [Brouwer and Heyting, 1975, p. 509]writes: "...intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of *move in time*, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the *empty form of the common substratum of all two-ities*. It is this common substratum, this empty form, which is the *basic intuition of mathematics*."

(23) This understanding of mathematical intuition is fundamentally different from Kant's or Gödel's [Gödel, 1964] understanding of intuition.

(24) The very creation of ideas is largely an unconscious process: "Most thought is unconscious ...inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes." [Lakoff and Núñez, 2000, p. 5].

about infinity as a paradigmatic example of an ideal mathematical element: "...nowhere is the infinite realized; it is neither present in nature nor admissible as a foundation in our rational thinking ...The role that remains to the infinite is, rather, merely that of an idea — if, in accordance with Kant's words, we understand by an idea a concept of reason that transcends all experience and through which the concrete is completed so as to form a totality ..." [Hilbert, 1926, p. 190]. In [Feferman, 2014, p. 4] Feferman writes: "The basic conceptions of mathematics are of certain kinds of relatively simple ideal world pictures that are not of objects in isolation but of structures" and "The basic objects of mathematical thought exist only as mental conceptions". For Hersh, mathematical objects are "mental objects with reproducible properties" [Hersh, 1997, p. 66].

I consider all the above descriptions of mathematical objects and concepts to be insufficiently clear because they refer to insufficiently clear psychological terms. The same applies to other descriptions of mathematics as a human invention, which I found in the literature. My view of the nature of mathematical models stems from my view of the essential role of language in thinking. My view of the essential role of language in thinking has its inspiration primarily in the works of von Humboldt [Humboldt, 1836] and Whorf [Carroll, 1956], and is explained in detail in [Čulina, 2021]. According to this view, language is not only a medium for expressing and communicating thoughts, but a medium in which thoughts are realized, a medium in which thoughts take their completed form. In the words of von Humboldt: "Language is the formative organ of thought. Intellectual activity, entirely mental, entirely internal, and to some extent passing without trace, becomes, through sound, externalized in speech and perceptible to the senses. Thought and language are therefore one and inseparable from each other. But the former is also intrinsically bound to the necessity of entering into a union with the verbal sound; thought cannot otherwise achieve clarity, nor the idea become a concept. The inseparable bonding of thought, vocal apparatus and hearing to language is unalterably rooted in the original constitution of human nature, which cannot be further explained ...without this transformation, occurring constantly with the help of language even in silence, into an objectivity that returns to the subject, the act of concept formation, and with it all true thinking, is impossible." [Humboldt, 1836, p. 50].

It is also important to point out here that, concerning thinking, the abstractions are the language abstractions, and not thinking about the so-called abstract objects. We talk about concrete objects (which can be real or imagined), and abstraction is achieved by extracting certain predicate and function symbols with which we talk about objects. For example, we count using concrete objects. Thus, the language of arithmetic talks about concrete objects (whose nature is not important to us), and with the language itself we achieve the appropriate abstraction. The language of arithmetic separates what is important to us when we use objects as numbers (first number, successor, predecessor, comparison, ...) from what is not important (e.g., size of marbles if we use collections of marbles for numbers, or font of decimal digits if we use sequences of decimal digits for numbers).

In this way of looking at the nature of thinking, which in its final effect is the creation and use of language, we can only realize mathematical models by building an appropriate mathematical language. By choosing names, function symbols and predicate symbols, we shape the initial intuition into a structured conception. Since the conception usually goes beyond our constructive capabilities, the constructed language has only partial interpretation, and that interpretation is internal — the interpretation in our internal world of activities. Since the interpretation is only partial, and because the imagined domain of interpretation is usually infinite, we cannot determine the truth values of all sentences of the language. Therefore, we must further specify the conception by using an appropriate set of axioms. Thus, the final mathematical model (theory) is a junction of axioms and partial internal interpretation of an adequate language. Sometimes, as we have seen on the example of the group theory, a mathematical model can be reduced to a set of axioms without an internal interpretation, although it arose from a corresponding intuition about the world of internal activities. Sometimes, the internal interpretation can be a total interpretation in another mathematical theory, as we have seen on the example of Euclidean geometry.

This way of looking at mathematical models can be acceptable to those who believe that mathematical concepts exist even without language. Given the concrete nature of language, unlike the vague nature of mental phenomena, they can accept mathematical language as a convenient representative of mental conceptions. I would also like to point out here that viewing abstract thinking as the creation and use of language does not mean denying the complex thought processes that take place behind it. This is a functional view: only the final effect of thinking is considered.

Furthermore, we must not forget that although a mathematical model is the final product of modelling an intuition about our internal world of activities, in real mathematical practice it is never isolated from the source from which it originated. This is especially important because a mathematical model, generally speaking, is not complete — there are multiple interpretations that are extensions of partial interpretation and that satisfy axioms; and intuition always leaves room for completion. In addition to testing a mathematical model as a means of rational cognition, it is also tested by how well it models the initial intuition, i.e. how well its consequences correspond to the intuition from which it emerged. In [Lakatos, 1976], Lakatos clearly demonstrated the importance for mathematics of this internal testing and revision of mathematical models (Lakatos calls this activity quasi-empirical mathematics). Feferman [Feferman, 2014] distinguishes mathematical conceptions by how close they are to everyday practice. The more distant they are, the less clear he considers them. For those conceptions that are completely clear, such as the conception of natural numbers, he believes that every statement within such a conception has a definite meaning, and thus has a truth value, regardless of whether we are able to determine its truth value or not, so the corresponding logic is the classical first-order logic. For those conceptions at the other end of clarity, such as the set theory, which lack some aspect of definiteness, he believes that the concept of truth may be partial and that the appropriate logic is semi-intuitionistic. Similarly, Ferreirós [Ferreirós, 2016] distinguishes mathematical conceptions according to whether their truth is based in our cognition and direct practice, such as the conception of natural numbers (Ferreirós then speaks of elementary mathematics characterized by certainty) or require additional hypotheses, such as the conception of the continuum (Ferreirós then talks about advanced mathematics, which is characterized by the presence of hypothetical statements — statements that gain or lose their objectivity through mathematical practice). The newer philosophy of mathematics, the so-called philosophy of mathematical practice, devotes special importance to these processes of interaction of mathematical models and basic intuitions, as one of the main sources of mathematics (see, for example, [Ferreirós, 2016; Mancosu, 2008]).

Although I consider it important how strongly mathematical models are connected to the internal world of activities, my view of mathematical models is more uniform. In my opinion, the basic criterion for evaluating mathematical models is their success as a tool of rational cognition. Seen in this way, there is no difference, for example, between the arithmetic and Hilbert spaces, although their connection to the internal world of activities is different. Also, in my opinion, there is no difference in the meaning of the truth of different mathematical models. To me, all mathematical statements are specifications, whether they specify what we will do with natural numbers in our internal world of activities, or whether they specify the truth of the continuum hypothesis. From such an understanding of the truth of mathematical concepts, it follows that the logic of mathematical models is classical logic, although perhaps in some situations, with regard to the problem that is being solved, it makes sense to look at mathematical models with non-classical logics.

Just as mathematical models are not isolated from the sources from which they arose, they are not isolated from each other either. Set theory is a natural environment for formulating and comparing mathematical models. In such an approach, axioms become the definition of a certain type of structure. However, set theory analyses the described structures in a uniform way, without going into their nature, whether they are extracted from the external world or from the internal world of our activities. Thus, although it gives an elegant mathematical description, set theory can also hide the true nature of mathematical models.

Mathematics is largely an elaborated language. The "magic" of mathematics is largely the "magic" of language. Inferring logical consequences from axioms, we establish what is true in a mathematical model. This can be very creative and exciting work and it seems that we discover truths about an existing exotic world, but we only unfold the specification. The key difference from scientific theories is that the interpretation here is in our internal world of activities and not in the external world. The external interpretation of a scientific theory enables us to test the theory, whether it has a power of rational cognition of nature. If the theory has such power then at least some objects of the theory exist in the primary sense of the word and at least some sentences of the theory are truths about nature. If the language does not have such a part, and that is the case with mathematics, then the objects we are talking about exist only within the conception (story), although they do not exist in the external world. Equally, if the language does not have an experimentally verifiable part then sentences we consider true within the conception are not true in the external world. We cannot experimentally verify that $|| + || = ||||$ $(2 + 2 = 4)$, not because it is an eternal truth of numbers, but because it is the way we add tallies. Likewise, we cannot experimentally verify that $(x^2)' = 2x$ because it is the consequence of how we imagined real numbers and functions. Mathematical objects are, possibly, through a partial internal interpretation, objects extracted from our internal world of activities, and mathematical truths are, possibly, through a partial internal interpretation, truths about our internal activities.(25) We are free to imagine any mathematical world. The real external existence of such a world is not important at all; all that matters is to be a successful thought tool in the process of rational cognition. In Cantor's words, "the essence of mathematics lies precisely in its freedom" [Cantor, 1883, p. 66]. The only constraint is, inside classical logic, that conceptions must not be contradictory. For Hilbert, in mathematics to exist means to be free of contradictions. In Hilbert's words: "the proof of the consistency of the axioms is at the same time the proof of the mathematical existence", [Hilbert, 1900, p. 265]. In Dedekind's words, "numbers are free creations of the human mind" [Dedekind, 1888, p. vii].

These views are in sharp contrast with historical views that mathematical truths exist really in some way and that we discover them and not create them. Historically, this change of view occurred in the 19th century with the appearance of non-Euclidean geometries. The new philosophical view of mathematics has freed the human mathematical powers and it has caused the blossoming of modern mathematics. It is a nice example of how philosophical views can influence science in a positive way. According to the old views mathematical truths are a particular kind of truths about the world. An exemplar is the Euclidean geometry — according to the

(25) Some statements get their truth through the partial interpretation of language, for example that 2+2 = 4, while for some statements the partial interpretation is not enough to determine their truth (e.g. the axiom of the completeness of real numbers or the continuum hypothesis). Then the assignment of truth to these statements is, at least in part (the axiom of the completeness of real numbers) or completely (the continuum hypothesis), a specification of an idea that transcends the internal world of activity.

old views, it discovers the truths about space. The appearance of non-Euclidean geometries, which are incompatible with Euclidean geometries but are equally logical in thinking and equally good candidates for the "true" geometry of the world, has definitely separated mathematics from the truths about nature. It has become clear that mathematics does not discover the truths about the world. If it discovers the truths at all they are at best the truths about our own activities in that world. From my personal teaching experience, I know that looking at mathematics as a free and creative human activity is a far better basis for learning mathematics than looking at it as an eternal truth about an elusive world. Claims that mathematical objects exist in the external world, real or special, and that mathematical truths are truths about such a world, are unfounded and lead to religion. Such a belief can of course be very inspiring and can produce very powerful mathematics, but mathematics itself is a human creation, in its final form a junction of axioms and a partial internal interpretation of an adequate language.

§ — Bibliography.

P. Apostoli, R. Hinnion, A. Kanda and T. Libert. Alternative set theories. In: **A. D. Irvine**, ed., *Philosophy of Mathematics*, pp. 461–491, North-Holland, Amsterdam, 2009.

P. W. Bridgman. *The Logic of Modern Physics*. Macmillan, 1927.

J. Bronowski. *The Ascent of Man*. Boston: Little, Brown, 1974.

L. Brouwer and A. Heyting. *L. E. J. Brouwer collected works: Philosophy and foundations of mathematics*. North-Holland, 1975.

G. Cantor. *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*. Teubner, Leipzig, 1883.

M. Carrier and J. Lenhard. *Mathematics as a Tool: Tracing New Roles of Mathematics in the Sciences*. Springer Verlag, 2017.

J. B. Carroll, ed.. *Language, Thought, and Reality: Selected Writings of Benjamin Lee Whorf*. The M.I.T. Press, 1956.

W. K. Clifford. *The Postulates of the Science of Space*. 1873. the third in a series of lectures Clifford delivered at the Royal Institution in London.

W. K. Clifford. *The Common Sense of the Exact Sciences*. Kegan, Paul, Trench, 1885.

B. Čulina. An Elementary System of Axioms for Euclidean Geometry Based on Symmetry Principles. *Axiomathes*, vol. 28, nº2, pp. 155–180, 2018.

B. Čulina. The Language Essence of Rational Cognition with Some Philosophical Consequences. *Tesis* (*Lima*), vol. 14, nº19, pp. 631–656, 2021.

B. Čulina. Mathematics - an Imagined Tool for Rational Cognition. `https://philpapers.org/archive/CULMA.pdf`, 2022. preprint.

B. Čulina. What is Mathematics for the Youngest? *Uzdanica*, vol. XIX, nºspecial issue, pp. 199–219, 2022.

R. Dedekind. *Stetigkeit und irrationale Zahlen*. Vieweg, 1872.

R. Dedekind. *Was sind und was sollen die Zahlen*. Vieweg, Braunschweig, 1888.

J. R. L. Delboeuf. *Prolégomènes Philosophiques de la Géométrie et Solution des Postulats*. J. Desoer, 1860.

M. Deveau and H. Teismann. $72 + 42$: Characterizations of the Completeness and Archimedean Properties of Ordered Fields. *Real Analysis Exchange*, vol. 39, nº2, pp. 261 – 304, 2014.

H. B. Enderton. *Elements of Set Theory*. Academic Press, 1977.

P. Ernest. *Social Constructivism as a Philosophy of Mathematics*. State University of New York Press, 1997.

S. Feferman. Logic, mathematics, and conceptual structuralism. In: **P. Rush**, ed., *The Metaphysics of Logic*, p. 72–92, Cambridge University Press, 2014.

J. E. Fenstad. *Structures and Algorithms: Mathematics and the Nature of Knowledge*. Springer Verlag, 2018.

J. Ferreirós. The Crisis in the Foundations of Mathematics. In: **T. Gowers**, ed., *Princeton Companion to Mathematics*, Princeton University Press, 2008.

J. Ferreirós. *Labyrinth of Thought. A History of Set Theory and its Role in Modern Mathematics*. Basel, Boston: Birkhäuser Verlag, 2. Ed., 2007.

J. Ferreirós. *Mathematical Knowledge and the Interplay of Practices*. Princeton University Press, 2016.

A. A. Fraenkel and Y. Bar-Hillel. *Foundations of Set Theory*. North-Holland, 1958.

K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. *Monatshefte für Mathematik und Physik*, vol. 38, pp. 173–198, 1931.

K. Gödel. What is Cantor's Continuum Problem (1964 Version). *Journal of Symbolic Logic*, n°2, pp. 116–117, 1964.

J. D. Hamkins. The Set-Theoretic Multiverse. *Review of Symbolic Logic*, vol. 5, n°3, pp. 416–449, 2012.

H. L. F. Helmholtz. Über die Tatsachen, die der Geometrie zu Grunde liegen. *Nachrichten der Königlichen Gesellschaft der Wissenschaften und der Georg - Augustus - Universität*, n°9, pp. 193 – 221, 1868.

R. Hersh. Some proposals for reviving the philosophy of mathematics. *Advances in Mathematics*, vol. 31, n°1, pp. 31–50, 1979.

R. Hersh. *What is Mathematics, Really?* Oxford University Press, 1997.

D. Hilbert. Mathematische Probleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, vol. 1900, pp. 253–297, 1900.

D. Hilbert. Über das Unendliche. *Mathematische Annalen*, vol. 95, pp. 161–190, 1926.

W. v. Humboldt. *Über die Verschiedenheit des menschlichen Sprachbaus und ihren Einfluss auf die geistige Entwicklung des Menschengeschlechts*. Bonn, F. Dümmler, 1836.

P. Kitcher. *The Nature of Mathematical Knowledge*. Oxford University Press, 1983.

I. Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge and London: Cambridge University Press, 1976.

G. Lakoff and R. Núñez. *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics Into Being*. Basic Books, 2000.

L. Löwenheim. Über Möglichkeiten im Relativkalkül. *Mathematische Annalen*, vol. 76, pp. 447–470, 1915.

S. Mac Lane. *Mathematics, Form and Function*. Springer-Verlag, 1986.

P. Mancosu, ed.. *The Philosophy of Mathematical Practice*. Oxford University Press, 2008.

J. Piaget. *Genetic Epistemology*. Columbia University Press, 1970.

J. H. Poincaré. *La Science et l'Hypothèse*. Flammarion, 1902.

G. F. B. Riemann. Über die Hypothesen, welche der Geometrie zu Grunde liegen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, vol. 13, 1883.

M. Schemmel. *Historical Epistemology of Space*. Springer, 2016.

W. Sieg. Hilbert's Programs: 1917?1922. *Bulletin of Symbolic Logic*, vol. 5, n°1, pp. 1–44, 1999.

T. Skolem. Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen. *Videnskapsselskapet Skrifter, I. Matematisk-naturvidenskabelig Klasse*, vol. 4, pp. 1–36, 1920.

A. Tarski. *Pojęcie prawdy w językach nauk dedukcyjnych*. Towarzystwo Naukowe Warszawskie, 1933.

R. Torretti. *Philosophy of geometry from Riemann to Poincaré*. Springer, 1978.

D. van Dalen. *Logic and Structure*. Springer Verlag, 5. Ed., 2013.

W. van Orman Quine. *Set Theory and its Logic*. Cambridge, MA, USA: Harvard University Press, 1963.

J. Wallis. *Opera Mathematica*. E Theatro Sheldoniano, 1695-1699. 3 Vols.

H. Weyl. *Raum, Zeit, Materie*. Julius Springer, 1918.

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