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A few notes on Bill Lawvere's Intellectual Biography[(*)]

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[(*)] The present text corresponds to the introductory speech I gave on August 3, 2024 for the memorial panel in honour of Bill Lawvere, which was held at the 25th World Congress of Philosophy in Rome.

One year ago, on January 23, William (Bill) Lawvere passed away at age 85. He was one of the greatest mathematicians of the 20th century and a pioneer of an original, dialectical, view of the foundations of mathematics once the language of category theory is exploited.

My eulogy will be devoted to providing basic information about his intellectual biography and emphasizing the main aspects of the turning point represented by Lawvere's research, which gave rise to a conceptual revolution in the philosophy of mathematics. For many years he also was my mentor, and best friend, thus, as you can easily understand, the few words I'm going to say will scratch only the surface of the intellectual debt I owe him.

Bill Lawvere was born in Muncie (Indiana) on February 9, 1937. He grew up in the country and subsequently helped his father in agricultural work. As he once told me, when he had to read a book, he would sit on the tractor and let him go straight for miles while absorbed by reading.

Lawvere completed his first-level university studies at the University of Indiana, graduating under the guidance of Clifford Truesdell, a unique character, who propounded the renaissance of rational continuum mechanics. At the time Lawvere was interested in physics more than mathematics, as he often wanted to remember, encouraging us to read what Truesdell had written. After graduation, Truesdell advised him to go to New York to meet Samuel Eilenberg, who in 1945 had introduced, in a joint paper with Saunders Many Lane, the very notion of mathematical "category", a notion very different from the traditional one, mainly tied to the legacy of Aristotle's and Kant's views, although, through type theory, the mathematical notion could serve as a tool to recover some aspects of those philosophical notions.

Eilenberg taught at Columbia University, where Lawvere earned his PhD in 1963 with a pioneering thesis. now recognised as one of historical importance, in which we can find

1. the invariant formulation, independent from the axiomatic "presentation" of algebraic theories, offering a new setting for universal algebra and introducing, in particular, the concept of "doctrine";

2. the first formulation of functorial semantics (the models of a theory are functors, with a decisive impact on analytic philosophy of language);

3. the categorical expression of the five Peano-Dedekind axioms for elementary arithmetic, by condensing them into a single axiom, which states the existence of a "natural number object" in a category;
4. the idea of the category of categories as a framework for all mathematics.

One year later, in 1964, Lawvere axiomatised into pure categorical language (without the use of $\in$) no less than set theory, something that, at the time, left Mac Lane astonished and puzzled many others.

In 1966 he went to Switzerland as a visiting professor at the Eidgenössische Technische Hochschule (ETH) in Zürich. It was in Zürich that he met Fatima Fenaroli. They married and since then Fatima has always been by his side, supporting him in hard times, because Lawvere's career was certainly not easy. Once back in America, he held positions at the University of Chicago (where Mac Lane was teaching as a professor of mathematics) and the Graduate Center of CUNY.

His revolutionary ideas in mathematics, and his radical marxism in philosophy and politics, set him at odds with the American academic establishment. The Canadian Dalhousie University, where he then taught, did not renew his contract because of his radically critical position towards the government. So Lawvere decided to move to the University of Perugia (Italy) from 1972 to 1974, where he held a seminar of great relevance thanks to which a small group of four young mathematicians formed around him, namely Renato Betti, Aurelio Carboni, Massimo Galuzzi, Gian Carlo Meloni, who would form the first nucleus of the Italian category-theorists. In 1974 he returned to the United States, at the State University of New York in Buffalo, where he remained as professor until retirement, except for an occasional period as a visiting professor in Paris in 1980-81.

Countless have been his lectures, as well as his research seminars, all over the world. The interest he was able to arouse in the listener was contagious. Among his students, I mention only two names for everyone, Marta Bunge and Anders Kock. The many young researchers who ventured into the territories explored by Lawvere have always found him open to their questions, and have received generous and collaborative support. It is thanks to him that various countries have been able to form research communities in the field of category theory: in addition to the United States and Canada, it has certainly happened in Italy (starting from the four who followed his seminars in Perugia), but also in Portugal (in Coimbra) and Colombia.

But let's come back to his work. After his doctoral thesis, his scientific output was vast: many are his research articles, next to which are many of his articles of a now panoramic, now programmatic (just think of the title of his 1973 work, "Continuously variable sets: algebraic geometry is geometry logic"), as well as articles on the philosophical meaning of category theory, and finally articles and books, of exemplary clarity, aimed at serving as an introduction to category theory.

He wrote two books of such a nature, destined to remain milestones: one with his dear friend Stephen Schanuel, *Conceptual Mathematics*, which had two editions, the first in 1997 and the second, widely revised, with a little help from me too, in 2011; and one, *Sets for Mathematics*, in collaboration with Bob Rosebrugh. Lawvere also edited (along with others) two volumes, which represent important stages for the links of category theory with logic and mathematical physics, respectively *Model Theory and Topoi* (1975) and *Categories in Continuum Physics.* On the website of the University of Buffalo, many of his works, divided by research area, are freely accessible in digital version.(1)

Lawvere has opened new paths, strictly tied to a philosophical meaning - not so much a meaning superimposed, as when a mathematician reflects on what he/she has done, but a meaning that had consciously oriented, in an essential way, mathematical research. Lawvere looked at mathematics and philosophy as directly connected within the framework of a dialectical conception of nature and thought, thus in a very different route from the mainstream in the philosophy of mathematics we were used to thinking from Frege and Russell onwards, according to which logic is supposed to have autonomous status and the universe of sets is supposed to be the only possible background.

More specifically, Lawvere's critical attitude against a widespread philosophical *understanding* of mathematics and of *doing* mathematics itself also had an intentional political value, so much to exert on him a sort of moral request. For what concerns the foundations

(1)Owing to the efforts of his son Danilo and his wife Fatima, a website entitled *The Lawvere Archives* now exists, entirely devoted to the work of Bill Lawvere. [note added 2025].

of mathematics, his emphasis on an irreducible contrast between a category-theoretic approach and the standard framework of logic and set theory has diminished in the last twenty years, with the recognition and ever-wider dissemination of his ideas and the need for a dialogue with a progressively wider community. The conference held in Florence in 2003, "Ramifications of Category Theory", represented a turning point, marking the desire to overcome misunderstandings and sterile contrast, and to establish a constructive confrontation with some proponents of the set-theoretic approach such as Dana Scott.

This does not mean Lawvere has, over the years, come up with compromises in contrast to his guiding ideas, among which, for instance, a new way of looking at Cantor's legacy. In fact, Lawvere urged a proper theory of sets as *Mengen*, as opposed to sets as *Kardinalen*. In the first sense, a set is a whole the parts of which are endowed of mutual cohesion, in addition to be variable rather than constant. Thus, only when cohesion is zeroed and variation is blocked — so that a set collapses to the dust of its points — we remain with sets in the second sense, i.e. with *Kardinalen*, whereas, in general, structure matters: not only it matters beyond number but also it has conceptual priority. Moreover, the two notions are related by means of left and right adjoints to the forgetful functor from a category of cohesive and variable sets to "standard" sets. Hence Lawvere's conviction that an algebraic approach (also developed by André Joyal) was more adequate and within this approach, issues concerning the size of a collection — just think of "large cardinals" — had marginal importance. Even the very notion of the infinite will lose, in his view, "ontological" relevance in favour of a structural perspective.

His dialectical vision of the foundations has always remained his firm point and although contemporary epistemology was for long time dismissed by Lawvere as a new sort of idealism and reactionary ideology, due to primary emphasis on the subjective side of knowledge, in the last twenty years his attitude opened up to a confrontation with some themes of philosophy of science and philosophy of language. We had projected to work on a joint paper presenting a categorical approach to epistemology, but our work was unfortunately made impossible by the pandemic. Nevertheless, as for a central question debated in the philosophy of language, such as the definition of "meaning", Lawvere 's attention was already witnessed by a brief observation he made in 1965 about the relationship, now expressible in categorical terms, between *Sinn* and *Bedeutung*, in Frege's lexicon, or *intension* and *extension*, in Carnap's lexicon.

In order to make possible such expression, and more generally in order to recognise the relevance of category-theoretic notions for epistemology and philosophy of language, a crucial step was taken by Lawvere himself with functorial semantics, as its introduction had by itself a decisive impact on model theory. But now models of a theory are also in categories other than **Sets** — and some theories have models *only* in categories different from **Sets**. Such a broadening in perspective is witnessed by synthetic-differential geometry, which originated by the intuitions of Lawvere and Kock, but it also makes it necessary, in logic, to reformulate the Completeness Theorem. Even the so-called "limitative theorems" proved by Gödel, Tarski and Turing acquire a new light in the framework of Cartesian closed categories. Moreover, categorial semantics is proper for higher-order typed languages and consequently relevant to theoretical computer science and cognitive sciences — see Lawvere's contribution to the volume *Logical Foundations of Cognition*, in 1994, edited by Gonzalo Reyes and John Macnamara.

One key point is that theories, in Lawvere's sense, *are* categories. Thus since scientific theories are theories, the view of models as functors from one category to another (not necessarily **Sets**) opened a new horizon also for the heated debates in philosophy of science.

Not surprisingly, for a long time, the above mentioned decisive impact of categorial methods on philosophy of language and epistemology was not recognised, due to the mathematical tools needed, which were clearly out of the mainstream, mostly bound to set-theoretic semantics. However, the times are changing, as evidenced by some works collected by Elaine Landry in the volume *Categories for the Working Philosopher* (2017), a title which goes hand in hand with that of the famous Mac Lane book of 1971, *Categories for the Working Mathematician*. The question is whether the recent interest in a "univalent" approach to foundations along the line proposed by Vladimir Voevodsky is compatible with how Lawvere conceived the categorial approach as endowed with a potential well beyond mathematics (and computer science), as his dialectical view was intended to cover the relationship between the mind and the world.

First, Lawvere never thought of category theory as the last word. Second, he was aware that the advance of knowledge is a collective enterprise of collaboration, and he often invited to put it concretely into practice, at the service of social progress (a conviction boldly manifested in the speech, "La guida alla natura", given on the occasion of the Giulio Preti Prize reception in 2011.[(2)] In particular, his "pedagogical" commitment was, to say the least, engaging and, in fact, he also said to me that such a commitment was an essential aspect of philosophy itself, which by consequence could not be reduced to just a critical, metalinguistic, analysis of language, knowledge, values and norms.

His general vision was moulded by the close, though very uncommon, connection he grasped between dialectical philosophy and the study of any mathematical structure in terms of maps (functors) from and to it, particularly through iterated pairs of adjoint functors, in terms of which the "unity of opposites" could finally find rigorous formulation. This connection led him to conceive a phenomenology of variable *Mannigfältigkeiten*, with different degrees of cohesion, somewhat in the style of Riemann but no longer confined to surfaces and higher-dimensional manifolds. Lawvere started focusing such phenomenology on the relationship between subject and object, mind and world, theories and models, the "abstract general" and the "concrete general", in the idea that dialectical materialism could be brought to new life. This was also a point we debated for years as such phenomenology does not necessarily need that philosophical framework, which, with analytic glasses, could be seen as no less problematic than the idealism he intended to oppose.

Daniel Kan, after having attended a seminar held by Eilenberg at the Columbia University, introduced the concept of adjoint functor in 1958. It was a big leap in the development of category theory, as it allowed to grasp in full generality the meaning of "universal constructions" and proved that category theory was more than a unifying *language*, useful to frame structural patterns in specific areas of mathematics, such as algebraic topology. But it was mainly after Alexander Grothendieck and others started using adjunctions in algebraic geometry that category theory turned out to be essential to *solve problems*. Since then the fertility of the new setting of category theory became clear, spreading through the most various fields of research, and particularly to theoretical computer science and logic.

[(2)] See the short video of the ceremony in Florence `www.youtube.com/watch?v=iXJCCyaH2Ks`.

Of course, this jump wasn't due to Lawvere alone and Lawvere himself regularly referred to the contribution provided by others to the specific subject he was exploring. He has always been generous in acknowledging debts to his "companions on the road" as well as drawing attention to ideas advanced in the past that had not received the attention they deserved. To make only one example of his interest in the recovery of approaches out of the mainstream, let me mention his categorical re-reading of Hermann Grasmann's *Ausdenungslehre*, published in 1844 (a year that Lawvere considered very important for another publication, also in German, by Marx and Engels).

Though Grothendieck had already exploited the notion of adjoint functor in algebraic geometry, it was Lawvere who gave the concept of adjunction that central position in foundations, which thus shifted from the identification of the primary components of the base of the mathematical building to a proper "architecture of mathematics", and Lawvere highlighted the specific *unifying* power provided by adjoints, with the aim of adhering to mathematical *practice*.

As for what concerns the philosophy of mathematics, the emphasis on architecture rather than just the basic building blocks was not a novelty. The same expression "architecture of mathematics" was the title of a programmatic article (1949) of Bourbaki's structuralism. In fact, the notion of adjoint functor would have been useful indeed to Bourbaki's general picture of mathematical structuralism, a picture supposed to be philosophically neutral, whereas Lawvere's materialistic view prevented him from sharing that pure, abstract, mathematical form could stay by itself. This difference is also relevant for the contemporary versions of categorial structuralism. More information on this point can be found in the papers collected in *Structures Mères: Semantics, Mathematics, and Cognitive Science* (2017).

The unifying power of adjunctions was clearly expressed by Lawvere in a 1969 article, "Adjointness in Foundations". If even Mac Lane came to say that «adjunctions are everywhere», it was thanks to that brilliant intuition by Lawvere, who also realised how to make it fruitful to rethink logic from scratch.

We can all agree that Lawvere was the father of categorial logic. True, there were also others who together with his pioneering works in this area. The three mathematicians who gave no less essential contributions and made possible present-day systematisation of categorical logic were Gonzalo Reyes, Jim Lambek and André Joyal. For what concerns specifically category-theoretic model theory we should add to these names also that of Michael Makkai. The "Montreal school" of algebraic logic was indeed a reality even before the use of the concept of category (just think of their investigations on polyadic algebras), but even in this case it was Lawvere's intuitions that traced the way to overcome obstacles to a satisfactory development of algebraic logic, first through the categorial treatment of connectives during the second half of the 1960s, in terms of Cartesian closed categories, and then, in 1969-70, through the definition of quantifiers as right and left adjoints of the substitution functor. As for the concept of topos, the concept comes from Grothendieck, "topos" being nothing more than an acronym for a generalised "*Topo*logical *s*pace", as Grothendieck named a category of sheaves varying over a base, which is no longer taken as a space defined in terms of its elements and its collection of opens but rather as a category purely described in terms of a suitable collection of maps which mimics the idea of a covering. However, the more general meaning of the term "topos" and its axiomatic reformulation is due to Lawvere and Myles Tierney, who in 1970 introduced the notion of "elementary topos" as a Cartesian closed category, equipped with a special object Ω as the *sub-object classifier*, and Grothendieck himself referred to Ω as the "Lawvere object" recognising the originality of Lawvere's contribution.

Unlike Grothendieck, he didn't get the Fields Medal, considered the Nobel Prize in mathematics. Let me recall that the Fields Medal is assigned "for outstanding discoveries in mathematics", and it generally goes to someone who has proved some important theorem that solves a long-outstanding problem and solves it in a way that also opens up new research horizons. Theorems, however, only exist because there is a conceptual framework that allows them to be formulated and an axiomatic theory in which they are proved. For what concerns problems related to foundations, higher-order constructive logic and theories that have no models in the universe of sets, such a conceptual framework and such a theory were found by Lawvere. Moreover, it's not that in his papers there are no theorems. There are, but they are not recognisable at first sight, as most of the time there is no tag "theorem", there is no tag "proof" and at the end of it there is no final square or q.e.d. This is possibly due to Lawvere's refusal, for many years, to format the exposition of his reasoning in the standard cliché, which undeniably remains useful. So everyone who has read one of his groundbreaking works has been busy identifying theorems and their proofs often using pen and paper to follow the formal, rapid, incisive, deep passages that crowded his pages and hosted results that, before you find them in his papers, would be difficult to imagine. For someone who didn't know what a genius was, meeting Lawvere would have made them understand.

This introduction is lacking in many respects, as I only pointed out some of the many traits of Lawvere's work. I hope that someone will face the task of providing a more detailed intellectual biography, duly emphasising the depth of the work of that great mathematician who was Bill Lawvere.

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